

SHARP LOCAL BOUNDEDNESS AND MAXIMUM PRINCIPLE IN THE INFINITELY DEGENERATE REGIME VIA DEGIORGI ITERATION

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ABSTRACT. We obtain local boundedness and maximum principles for weak subsolutions to certain infinitely degenerate elliptic divergence form equations using the DeGiorgi method. For example, we show that if $f(r) \geq e^{-r^{-\sigma}}$ for some $0 < \sigma < 1$, then all weak subsolutions to the associated infinitely degenerate quasilinear equations of the form

$$\operatorname{div} \mathcal{A}(x, u) \operatorname{grad} u = \phi(x, y), \quad \mathcal{A}(x, z) \sim \begin{bmatrix} I_{n-1} & 0 \\ 0 & f(x_1)^2 \end{bmatrix},$$

with rough data \mathcal{A} and A -admissible ϕ , are locally bounded. Examples from our earlier paper [KoRiSaSh] show there are locally unbounded weak solutions when $\sigma > 1$, and so the above result eliminates the ‘Moser gap’ left open in [KoRiSaSh]. Finally we obtain a maximum principle for weak subsolutions under a very mild condition on the degeneracy function $f(x)$, essentially that $\ln f(x)$ is merely doubling on $(0, \infty)$.

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1. INTRODUCTION

In [KoRiSaSh], we established local boundedness and maximum principles for weak subsolutions to certain infinitely degenerate elliptic divergence form equations, motivated by the pioneering work of Fedii [Fe], Kusuoka and Strook [KuStr], Morimoto [Mor] and Christ [Chr]. For example, we considered the family $\{f_{k,\sigma}\}_{k \in \mathbb{N}, \sigma > 0}$ with

$$f_{k,\sigma}(x) = |x|^{(\ln^{(k)} \frac{1}{|x|})^\sigma}, \quad -\infty < x < \infty,$$

of infinitely degenerate functions at the origin, and showed that if **either** $k \geq 2$ and $0 < \sigma < \infty$, **or** $k = 1$ and $0 < \sigma < 1$, then all weak subsolutions to the associated infinitely degenerate quasilinear equations of the form

$$\mathcal{L}u \equiv \operatorname{div} \mathcal{A}(x, y, u) \nabla u = \phi(x, y), \quad \mathcal{A}(x, y, z) \sim \begin{bmatrix} 1 & 0 \\ 0 & f_{k,\sigma}(x)^2 \end{bmatrix},$$

with rough data \mathcal{A} and A -admissible ϕ , are locally bounded. We also obtained a 3-dimensional analogue of this result, as well as a maximum principle in 2 and 3 dimensions for weak subsolutions under the same restrictions on k and σ .

The key to these results was an Orlicz-Sobolev bump inequality

$$\Phi^{(-1)} \left(\int_B \Phi(|u|) d\mu_B \right) \leq C \varphi(r(B)) \|\nabla_A u\|_{L^1(\mu_B)}, \quad u \in Lip_{\text{compact}}(B),$$

with degenerate bump function $\Phi(t) = \Phi_m(t) = e^{[(\ln t)^{\frac{1}{m}} + 1]^m}$ in place of the usual power bumps $\Phi(t) = t^\sigma$ for $\sigma > 0$. Then we combined this Orlicz-Sobolev bump inequality with the usual Cacciopoli inequality

$$\int_B \psi_B^2 \|\nabla_A u\|^2 \leq C \int_B u^2 \|\nabla_A \psi_B\|^2,$$

using the method of *Moser iteration* [Mos]. The result was the familiar sequence of bootstrap inequalities of the form

$$(1.1) \quad \Phi^{(-1)} \left(\frac{1}{\gamma_n} \int_{B(0, r_{n+1})} \Phi(|f_n(u)|^2) d\mu_{r_{n+1}} \right) \leq C_n \int_{B(0, r_n)} |f_n(u)|^2 d\mu_{r_n},$$

where the balls $B(0, r_n)$ shrink to a ball $B(0, r_\infty)$ with $r_\infty > 0$, whenever $f_n(u)$ is a subsolution to the linear equation $Lu = \text{div } A(x, y) \nabla u = \phi$ (see [KoRiSaSh] for details).

However, an essential restriction on the method of Moser iteration arose when the bootstrap inequalities (1.1) were iterated - namely the requirement that the bump function Φ in the Orlicz-Sobolev bump inequality be Φ_m as above with $m > 2$. This placed a limit on the degeneracy of our geometry which was reflected in our assumption that $f = f_{k, \sigma}$ with $k = 1$ and $0 < \sigma < 1$. With this limit on the degeneracy of the geometry, our local boundedness result was not sharp in either 2 or 3 dimensions, missing in dimension 3 by an entire iterated log due to the example in Theorem 115 of [KoRiSaSh], which showed that unbounded solutions can exist for geometries $F_{0, \sigma} = -\ln f_{0, \sigma}$ with $\sigma > 1$ (see Notation 1 below for an explanation of our use of the term ‘geometry’ in this context), while we were unable to produce any counterexamples at all for such equations in the plane.

The purpose of this paper is to rectify this lack of sharpness in dimensions $n \geq 3$ by showing that if we use the truncation method of *DeGiorgi iteration*, first presented in [DeG], to combine Orlicz-Sobolev bump inequalities with the Cacciopoli inequalities, then we can prove local boundedness in the gap left open by Moser iteration, namely the gap from $F_{0, 1+\varepsilon}$ to $F_{1, 1-\varepsilon}$. We further improve matters by showing that this local boundedness holds for merely a *one-sided* degeneracy inequality, as well as in higher dimensions.

It might be useful for the reader to keep in mind the following expanded scale of degenerate geometries parameterized by the function $F = -\ln f$:

$$\begin{aligned} D_\sigma(r) &\equiv \left(\frac{1}{r}\right)^\sigma, \quad \sigma > 0, \\ F_{k, \sigma}(r) &\equiv \left(\ln \frac{1}{r}\right) \left(\ln^{(k)} \frac{1}{r}\right)^\sigma, \quad k \geq 1, \sigma > 0, \\ H_N(r) &\equiv N \ln \frac{1}{r}, \quad N \geq 0, \end{aligned}$$

that satisfy

$$H_{N_1}(r) \lesssim H_{N_2}(r) \lesssim F_{k_1, \sigma_1}(r) \lesssim F_{k_1, \sigma_2}(r) \lesssim F_{k_2, \sigma_1}(r) \lesssim F_{k_2, \sigma_2}(r) \lesssim D_{\sigma_1}(r) \lesssim D_{\sigma_2},$$

provided $N_1 \leq N_2$, $k_1 > k_2$ or $k_1 = k_2$ and $\sigma_1 \leq \sigma_2$. Thus the geometry $H_0(r) = 0$ corresponds to the elliptic Euclidean geometry, $H_N(r)$ corresponds to the finite type N geometries, $F_{k, \sigma}(r)$ corresponds to a near finite type geometry that drifts further from finite type as k decreases and σ increases, and $D_\sigma(r)$ corresponds to a strongly degenerate geometry whose degeneracy increases with σ . Note that if we formally set $k = 0$ in the definition of $F_{k, \sigma}$ we obtain

$$\begin{aligned} F_{0, \sigma}(r) &= \left(\ln \frac{1}{r}\right) \left(\frac{1}{r}\right)^\sigma = \left(\ln \frac{1}{r}\right) D_\sigma(r) \\ \implies D_\sigma(r) &\leq F_{0, \sigma}(r) \leq C_\varepsilon D_{\sigma+\varepsilon}(r), \quad \varepsilon > 0, \end{aligned}$$

so that we may effectively interchange $F_{0, \sigma}$ and D_σ .

1.1. Main results. In this paper we show that all weak subsolutions to the infinitely degenerate quasilinear equations $\mathcal{L}u = \phi$ with rough data \mathcal{A} and A -admissible ϕ , are locally bounded provided the geometry F associated with A has degeneracy no worse than D_σ for some $0 < \sigma < 1$. More precisely, we show local boundedness of subsolutions for geometries F with $F \leq D_\sigma$ on $(0, \infty)$, and $0 < \sigma < 1$. This improves the local boundedness result in [KoRiSaSh], where we assumed $F_{1,\sigma}$ instead of $F_{0,\sigma} \sim D_\sigma$, restricted attention to dimensions 2 and 3 only, and where we also assumed $F = F_{1,\sigma}$ rather than just a one-sided degeneracy inequality.

Most importantly, we now obtain that this local boundedness result in dimension $n \geq 3$ is *sharp* using Theorem 115 in [KoRiSaSh], that shows that local boundedness of solutions can fail for the geometries D_σ with $\sigma > 1$, building on work of Kusuoka and Strook [KuStr] and Morimoto [Mor]. Thus DeGiorgi iteration captures the gap left open by Moser iteration, and points to an improved robustness of DeGiorgi's method as compared to that of Moser, something not immediately evident in the elliptic regime. See however Caffarelli and Vasseur [CaVa] and references given there for applications of DeGiorgi's truncation method.

Finally, we obtain a maximum principle for weak subsolutions under very mild conditions on the degeneracy function f , namely that $F = -\ln f$ satisfies the five structure conditions in Definition 7 below, which roughly speaking amount to doubling assumptions on $F = -\ln f$ and F' . Here are the precise statements. Definitions are given in the following subsection.

Theorem 1. *Suppose that $D \subset \mathbb{R}^n$ is a domain in \mathbb{R}^n with $n \geq 2$ and that*

$$\mathcal{L}u \equiv \operatorname{div} \mathcal{A}(x, u) \nabla u, \quad x = (x_1, \dots, x_n) \in D,$$

where $\mathcal{A}(x, z) \sim \begin{bmatrix} I_{n-1} & 0 \\ 0 & f(x_1)^2 \end{bmatrix}$, I_{n-1} is the $(n-1) \times (n-1)$ identity matrix, \mathcal{A} has bounded measurable components, and the geometry $F = -\ln f$ satisfies the structure conditions in Definition 7 below.

- (1) *If $F \leq D_\sigma$ for some $0 < \sigma < 1$, then every weak subsolution to $\mathcal{L}u = \phi$ with A -admissible ϕ is locally bounded in D .*
- (2) *On the other hand, if $n \geq 3$ and $\sigma > 1$, then there exists an unbounded weak solution u in a neighbourhood of the origin in \mathbb{R}^n to the equation $\mathcal{L}u = 0$ with geometry $F = D_\sigma$.*

Theorem 2. *Suppose that F satisfies the structure conditions in Definition 7. Assume that u is a weak subsolution to $\mathcal{L}u = \phi$ in a domain $\Omega \subset \mathbb{R}^n$ with $n \geq 3$, where \mathcal{L} has degeneracy F and ϕ is A -admissible. Moreover, suppose that u is bounded in the weak sense on the boundary $\partial\Omega$. Then u is globally bounded in Ω and satisfies*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \|\phi\|_{X(\Omega)}.$$

The proofs of both theorems are completed in the final section of this paper.

1.2. Preliminaries and definitions. Before beginning the proofs of these theorems using DeGiorgi's method of iterating Sobolev and Cacciopoli inequalities, we recall some of the terminology and definitions from [KoRiSaSh] that we use here.

Let $A(x)$ be a nonnegative semidefinite $n \times n$ matrix valued function in a bounded domain $\Omega \subset \mathbb{R}^n$. We consider the second order *special* quasilinear equation ('special' because only u , and not ∇u , appears nonlinearly),

$$(1.2) \quad \mathcal{L}u \equiv \nabla^{\operatorname{tr}} \mathcal{A}(x, u(x)) \nabla u = \phi, \quad x \in \Omega,$$

and we assume the following quadratic form condition on the quasilinear matrix $\mathcal{A}(x, u(x))$,

$$(1.3) \quad k \xi^T A(x) \xi \leq \xi^T \mathcal{A}(x, z) \xi \leq K \xi^T A(x) \xi,$$

for a.e. $x \in \Omega$ and all $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$. Here k, K are positive constants and we assume that $A(x) = B(x)^{\operatorname{tr}} B(x)$ where $B(x)$ is a Lipschitz continuous $n \times n$ real-valued matrix defined for $x \in \Omega$. We also consider the linear equation

$$Lu \equiv \nabla^{\operatorname{tr}} A(x) \nabla u = \phi, \quad x \in \Omega,$$

and define the A -gradient by

$$(1.4) \quad \nabla_A = B(x) \nabla.$$

Definition 3. The degenerate Sobolev space $W_A^{1,2}(\Omega)$ is normed by

$$\|v\|_{W_A^{1,2}} \equiv \sqrt{\int_{\Omega} (|v|^2 + \nabla v^{\text{tr}} A \nabla v)} = \sqrt{\int_{\Omega} (|v|^2 + |\nabla_A v|^2)}.$$

Definition 4. Let Ω be a bounded domain in \mathbb{R}^n . Assume that $\phi \in L_{\text{loc}}^2(\Omega)$. We say that $u \in W_A^{1,2}(\Omega)$ is a weak solution to $\mathcal{L}u = \phi$ provided

$$-\int_{\Omega} \nabla w(x)^{\text{tr}} \mathcal{A}(x, u(x)) \nabla u = \int_{\Omega} \phi w$$

for all $w \in (W_A^{1,2})_0(\Omega)$, where $(W_A^{1,2})_0(\Omega)$ denotes the closure in $W_A^{1,2}(\Omega)$ of the subspace of Lipschitz continuous functions with compact support in Ω .

Note that our quadratic form condition (1.3) implies that the integral on the left above is absolutely convergent, and our assumption that $\phi \in L_{\text{loc}}^2(\Omega)$ implies that the integral on the right above is absolutely convergent. Weak sub and super solutions are defined by replacing $=$ with \geq and \leq respectively in the display above.

Given a geometry $F = -\ln f$, we define the balls B to be the control balls associated with the $n \times n$ matrix $\begin{bmatrix} I_{n-1} & 0 \\ 0 & f(x_1)^2 \end{bmatrix}$ (see e.g. [KoRiSaSh]). Assuming the structure conditions in Definition 7 below, we recall from [KoRiSaSh] that the Lebesgue measure of the *two* dimensional ball $B_{2D}(x, r)$ centered at $x \in \mathbb{R}^2$ with radius $r > 0$ satisfies

$$(1.5) \quad |B_{2D}(x, r)| \approx \begin{cases} r^2 f(x_1) & \text{if } r \leq \frac{1}{|F'(x_1)|} \\ \frac{f(x_1+r)}{|F'(x_1+r)|^2} & \text{if } r \geq \frac{1}{|F'(x_1)|} \end{cases}.$$

Definition 5 (Standard sequence of accumulating Lipschitz functions). Let Ω be a bounded domain in \mathbb{R}^n and let $A(x)$ be a nonnegative semidefinite $n \times n$ matrix valued function as above. Let $\gamma > 1$. Fix $r > 0$ and $x \in \Omega$. We define an (A, d) -standard sequence of Lipschitz cutoff functions $\{\psi_j\}_{j=1}^{\infty}$ at (x, r) , associated with sets $B(x, r_j) \supset \text{supp } \psi_j$, to be a sequence satisfying $\psi_j = 1$ on $B(x, r_{j+1})$, $r_1 = r$, $r_{\infty} \equiv \lim_{j \rightarrow \infty} r_j = \frac{1}{2}$, $r_j - r_{j+1} = \frac{c}{j^{\gamma}} r$ for a uniquely determined constant $c = c_{\gamma}$ depending on $\gamma > 1$, and $\|\nabla_A \psi_j\|_{\infty} \lesssim \frac{j^{\gamma}}{r}$ with ∇_A as above.

Definition 6. Let Ω be a bounded domain in \mathbb{R}^n and let $A(x)$ be a nonnegative semidefinite $n \times n$ matrix valued function as above. Fix $x \in \Omega$ and $\rho > 0$. We say ϕ is A -admissible at (x, ρ) if

$$\|\phi\|_{X(B(x, \rho))} \equiv \sup_{v \in (W_A^{1,1})_0(B(x, \rho))} \frac{\int_{B(x, \rho)} |v \phi| dy}{\int_{B(x, \rho)} \|\nabla_A v\| dy} < \infty.$$

Definition 7 (structure conditions). We refer to the following five conditions on $F : (0, \infty) \rightarrow \mathbb{R}$ as structure conditions:

- (1) $\lim_{x \rightarrow 0^+} F(x) = +\infty$;
- (2) $F'(x) < 0$ and $F''(x) > 0$ for all $x \in (0, R)$;
- (3) $\frac{1}{C} |F'(r)| \leq |F'(x)| \leq C |F'(r)|$ for $\frac{1}{2}r < x < 2r < R$;
- (4) $\frac{1}{-x F'(x)}$ is increasing in the interval $(0, R)$ and satisfies $\frac{1}{-x F'(x)} \leq \frac{1}{\varepsilon}$ for $x \in (0, R)$;
- (5) $\frac{F''(x)}{-F'(x)} \approx \frac{1}{x}$ for $x \in (0, R)$.

Remark 8. We make no smoothness assumption on f other than the existence of the second derivative f'' on the open interval $(0, R)$. Note also that at one extreme, f can be of finite type, namely $f(x) = x^{\alpha}$ for any $\alpha > 0$, and at the other extreme, f can be of strongly degenerate type, namely $f(x) = e^{-\frac{1}{x^{\alpha}}}$ for any $\alpha > 0$. Assumption (1) rules out the elliptic case $f(0) > 0$.

Using these structure conditions, we can show that standard sequences of Lipschitz cutoff functions always exist for our geometries.

Lemma 9. *If $\gamma > 1$ and $A(x)$ is a continuous nonnegative semidefinite $n \times n$ matrix valued function on a bounded domain $\Omega \subset \mathbb{R}^n$ as above, and if d is the associated control metric, then for every $r > 0$ and $x \in \Omega$, there is an (A, d) -standard sequence of Lipschitz cutoff functions $\{\psi_j\}_{j=1}^\infty$ at (x, r) , associated with balls $B(x, r_j) \supset \text{supp } \psi_j$.*

Proof. This follows immediately from Proposition 68 on page 90 of [SaWh4], once we observe that in the proof of Proposition 68, we can take N to be any real number greater than 1 (so that $\sum_{j=1}^\infty j^{-N} < \infty$), and that the assumption of the containment condition in Proposition 68 there was *only* used in the proof to conclude that the annuli $B(x, t) \setminus B(x, s)$ have positive Euclidean thickness for $0 < s < t < \infty$ - i.e. that the boundaries $\partial B(x, t)$ and $\partial B(x, s)$ are pairwise disjoint. This is certainly the case for the control balls $B(x, r)$ associated with our geometries F satisfying Definition 7, and so the proof of Proposition 68 of [SaWh4] applies to prove Lemma 9. \square

Definition 10. *We say that $u \in W_A^{1,2}(\Omega)$ is nonpositive in the weak sense on the boundary $\partial\Omega$ of a domain Ω provided that $u^+ \equiv \max\{u, 0\} \in (W_A^{1,2})_0(\Omega)$. More generally, we say $u \leq \ell$ in the weak sense on $\partial\Omega$ if $u - \ell$ is nonpositive in the weak sense on the boundary $\partial\Omega$. Finally we set*

$$\sup_{\partial\Omega} u = \inf \left\{ \ell \in \mathbb{R} : (u - \ell)^+ \equiv \max\{u - \ell, 0\} \in (W_A^{1,2})_0(\Omega) \right\}.$$

Notation 11. *We refer to a function F satisfying the structure conditions in Definition 7 as a ‘geometry’ since $F = -\ln f$ then specifies the nonnegative semidefinite matrix $M_F = \begin{bmatrix} I_{n-1} & 0 \\ 0 & f(x_1)^2 \end{bmatrix}$ and hence the geometry of the associated control balls. The class of degenerate elliptic linear operators*

$$Lu = \text{div } A(x) \nabla u, \quad A(x) \sim M_F(x_1),$$

is also specified along with the associated class of quasilinear operators

$$\mathcal{L}u = \text{div } A \nabla u, \quad A(x, z) \sim M_F(x_1).$$

2. DEGIORGI ITERATION

Let Φ be a Young function on $(0, \infty)$ and let F be a geometry satisfying the structure conditions in Definition 7. We will assume initially that we have an Orlicz-Sobolev norm inequality for the control balls B is some domain $\Omega \subset \mathbb{R}^n$:

$$(2.1) \quad \|w\|_{L^\Phi(B)} \leq \varphi(r(B)) \|\nabla_A w\|_{L^1(B)}, \quad w \in (W_A^{1,2})_0(B),$$

for some ‘superradius’ function $\varphi(r) \geq r$, $0 < r < \infty$, and later prove this for appropriate geometries. Recall that if u is a weak subsolution to $\mathcal{L}u = \phi$ with A -admissible ϕ , then we also have the standard Cacciopoli inequality for u_+ on a ball B with Lipschitz ψ supported in B :

$$(2.2) \quad \int_B |\nabla_A(\psi u_+)|^2 \leq C (\|\psi\|_{L^\infty} + \|\nabla_A \psi\|_{L^\infty})^2 \int_B (u_+ + \|\phi\|_X)^2.$$

Indeed, $\nabla A \nabla u = \phi$ in the weak sense implies

$$\begin{aligned} \int (\psi^2 u_+) \phi &= \int (\psi^2 u_+) \nabla A \nabla u = - \int \nabla (\psi^2 u_+) A \nabla u_+ \\ &= - \int 2\psi u_+ (\nabla \psi) A \nabla u_+ - \int \psi^2 (\nabla u_+) A \nabla u_+, \end{aligned}$$

which gives

$$\begin{aligned} \int \psi^2 |\nabla_A u_+|^2 &= - \int 2\psi u_+ (\nabla \psi) A \nabla u_+ - \int (\psi^2 u_+) \phi \\ &\leq 2 \left(\int |\psi \nabla_A u_+|^2 \right)^{\frac{1}{2}} \left(\int |\nabla_A \psi|^2 u_+^2 \right)^{\frac{1}{2}} + \|\phi\|_X \int |\nabla_A (\psi^2 u_+)|. \end{aligned}$$

Now

$$\begin{aligned} \int |\nabla_A(\psi^2 u_+)| &= \int |u_+ \nabla_A \psi^2 + \psi^2 \nabla_A u_+| \leq \int |u_+ \nabla_A \psi^2| + \left(\int |\psi \nabla_A u_+|^2 \right)^{\frac{1}{2}} \left(\int |\psi|^2 \right)^{\frac{1}{2}} \\ &\leq \|\nabla_A \psi^2\|_{L^\infty} \int_B |u_+| + \|\psi\|_{L^\infty} \sqrt{|B|} \left(\int |\psi \nabla_A u_+|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

and so altogether we have

$$\begin{aligned} \int |\psi \nabla_A u_+|^2 &\lesssim \left(\int |\psi \nabla_A u_+|^2 \right)^{\frac{1}{2}} \left\{ \left(\int |\nabla_A \psi|^2 u_+^2 \right)^{\frac{1}{2}} + \|\phi\|_X \|\psi\|_{L^\infty} \sqrt{|B|} \right\} \\ &\quad + \|\phi\|_X \|\nabla_A \psi^2\|_{L^\infty} \int_B |u_+|, \end{aligned}$$

which is easily seen to imply, upon considering the larger of the two summands on the right hand side above,

$$\begin{aligned} \left(\int |\psi \nabla_A u_+|^2 \right)^{\frac{1}{2}} &\lesssim \left(\int |\nabla_A \psi|^2 u_+^2 \right)^{\frac{1}{2}} + \|\phi\|_X \|\psi\|_{L^\infty} \sqrt{|B|} + \sqrt{\|\phi\|_X \|\nabla_A \psi^2\|_{L^\infty} \int_B |u_+|} \\ &\lesssim \|\nabla_A \psi\|_{L^\infty} \left(\int_B u_+^2 \right)^{\frac{1}{2}} + \|\phi\|_X \|\psi\|_{L^\infty} \sqrt{|B|} + \sqrt{\|\phi\|_X \|\nabla_A \psi\|_{L^\infty} \|\psi\|_{L^\infty} \sqrt{|B|} \left(\int_B |u_+|^2 \right)^{\frac{1}{2}}} \\ &\lesssim (\|\psi\|_{L^\infty} + \|\nabla_A \psi\|_{L^\infty}) \left(\int_B (u_+ + \|\phi\|_X)^2 \right)^{\frac{1}{2}} + \|\nabla_A \psi^2\|_{L^\infty} \|\phi\|_X \sqrt{|B|} + \|\psi\|_{L^\infty} \left(\int_B |u_+|^2 \right)^{\frac{1}{2}} \\ &\lesssim (\|\psi\|_{L^\infty} + \|\nabla_A \psi\|_{L^\infty}) \left(\int_B (u_+ + \|\phi\|_X)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

We now use

$$\int_B |\nabla_A(\psi u_+)|^2 = \int_B |u_+ \nabla_A \psi + \psi \nabla_A u_+|^2 \leq 2 \left\{ \int_B |u_+ \nabla_A \psi|^2 + \int_B |\psi \nabla_A u_+|^2 \right\}$$

to complete the proof of the Cacciopoli inequality (2.2).

In the next proposition we apply DeGiorgi iteration to a sequence of Orlicz-Sobolev and Cacciopoli inequalities involving a family of bump functions adapted to the strongly degenerate geometries D_σ . It is important to note that (2.2) holds for u_+ whenever u is a weak solution *without* assuming that u_+ is also a subsolution. More precisely, recall that the strongly degenerate geometries D_α have degeneracy function

$$f(x) = e^{-\frac{1}{x^\alpha}}, \quad \alpha > 0, \quad x \geq 0.$$

The appropriate family of corresponding bump functions is given by

$$(2.3) \quad \Phi_N(t) = e^{\ln t + N \ln \ln t}, \quad t \text{ large}, \quad N \geq 1.$$

Proposition 12. *Assume that the Orlicz-Sobolev norm inequality (2.1) holds with $\Phi = \Phi_N$ for some $N > 1$ and a geometry F satisfying Definition 7. Then every weak subsolution u to $\mathcal{L}u = \phi$ in Ω , with A -admissible ϕ , satisfies the inner ball inequality*

$$\|u_+ + \|\phi\|_{X(B)}\|_{L^\infty(\frac{1}{2}B)} \leq C_N \|u_+ + \|\phi\|_{X(B)}\|_{L^2(B)},$$

for every ball $B \subset \Omega$ that is centered on the x_n -axis, and hence u is locally bounded above in Ω since \mathcal{L} is elliptic away from the x_n -axis by the structure conditions in Definition 7. In particular, weak solutions are locally bounded in Ω .

Proof. Without loss of generality, we may assume that $B = B(0, r)$ is a ball centered at the origin with radius $r > 0$. Let $\{\psi_j\}_{j=1}^\infty$ be a standard sequence of Lipschitz cutoff functions at $(0, r)$ as in Lemma 9 with $\gamma = 1 + \frac{\varepsilon}{2}$, and associated with the balls $B_j \equiv B(0, r_j) \supset \text{supp } \psi_{j+1}$, $\psi_j = 1$ on B_j , with $r_1 = r$, $r_\infty \equiv \lim_{j \rightarrow \infty} r_j = \frac{1}{2}$, $r_j - r_{j+1} = \frac{c}{j^{1+\frac{\varepsilon}{2}}} r$ for a uniquely determined constant $c = c_\varepsilon$, and $\|\nabla_A \psi_j\|_\infty \lesssim \frac{j^{1+\frac{\varepsilon}{2}}}{r}$

with ∇_A as in (1.4) above (see Proposition 68 in [SaWh4] for more detail). Following DeGiorgi ([DeG], see also [CaVa]), we consider the family of truncations

$$u_k = (u - C_k)_+, \quad C_k = 1 - c(k+1)^{-\varepsilon/2},$$

and denote the L^2 norm of the truncation u_k by

$$U_k \equiv \int_{B_k} |u_k|^2 dx.$$

Using Hölder's inequality for Young functions we can write

$$(2.4) \quad \int (\psi_{k+1} u_{k+1})^2 \leq C \|(\psi_{k+1} u_{k+1})^2\|_{L^\Phi(B_k)} \cdot \|1\|_{L^{\tilde{\Phi}}(\{\psi_{k+1} u_{k+1} > 0\})}.$$

For the first factor on the right we have, using the Orlicz-Sobolev inequality (2.1),

$$\|(\psi_{k+1} u_{k+1})^2\|_{L^\Phi(B_k)} \leq \int |\nabla_A (\psi_{k+1}^2 u_{k+1}^2)| \leq \delta \int |\nabla_A (\psi_{k+1} u_{k+1})|^2 + \frac{1}{\delta} \int (\psi_{k+1}^2 u_{k+1}^2),$$

and by the Cacciopoli inequality (2.2) we have,

$$\begin{aligned} \int |\nabla_A (\psi_{k+1} u_{k+1})|^2 &\leq C (\|\nabla_A \psi_{k+1}\|_{L^\infty} + \|\psi_{k+1}\|_{L^\infty})^2 \int_{B_k} u_{k+1}^2 \\ &\leq C (k+1)^{2+\varepsilon} \int_{B_k} u_{k+1}^2. \end{aligned}$$

Finally, choosing $\delta = \frac{1}{(k+1)^{1+\frac{\varepsilon}{2}}}$, we obtain

$$(2.5) \quad \|(\psi_{k+1} u_{k+1})^2\|_{L^\Phi(B_k)} \leq C (k+1)^{1+\frac{\varepsilon}{2}} \int_{B_k} u_{k+1}^2.$$

For the second factor we claim

$$(2.6) \quad \|1\|_{L^{\tilde{\Phi}}(\{\psi_{k+1} u_{k+1} > 0\})} \leq \tilde{\Psi}^{-1} \left(\left| \left\{ \psi_k u_k > \frac{\varepsilon}{2} c (k+2)^{-1-\frac{\varepsilon}{2}} \right\} \right| \right),$$

with the notation

$$(2.7) \quad \tilde{\Psi}^{-1}(t) \equiv \frac{1}{\tilde{\Phi}^{-1}\left(\frac{1}{t}\right)}.$$

First recall

$$\|f\|_{L^{\tilde{\Phi}}(X)} \equiv \inf \left\{ a : \int_X \tilde{\Phi} \left(\frac{f}{a} \right) \leq 1 \right\},$$

and note

$$\int_{\{\psi_{k+1} u_{k+1} > 0\}} \tilde{\Phi} \left(\frac{1}{a} \right) = \tilde{\Phi} \left(\frac{1}{a} \right) |\{\psi_{k+1} u_{k+1} > 0\}|.$$

Now take

$$a = \tilde{\Psi}^{-1}(|\{\psi_{k+1} u_{k+1} > 0\}|) \equiv \frac{1}{\tilde{\Phi}^{-1}\left(\frac{1}{|\{\psi_{k+1} u_{k+1} > 0\}|}\right)}$$

which obviously satisfies

$$\int_{\{\psi_{k+1} u_{k+1} > 0\}} \tilde{\Phi} \left(\frac{1}{a} \right) = 1.$$

This gives

$$\|1\|_{L^{\tilde{\Phi}}(\{\psi_{k+1} u_{k+1} > 0\})} \leq a = \tilde{\Psi}^{-1}(|\{\psi_{k+1} u_{k+1} > 0\}|),$$

and to conclude (2.6) we only need to observe that

$$\{\psi_{k+1} u_{k+1} > 0\} \subset \left\{ \psi_{k+1} u_k > \frac{\varepsilon}{2} c (k+2)^{-1-\frac{\varepsilon}{2}} \right\},$$

since

$$\begin{aligned} u_{k+1} &> 0 \implies u > c_{k+1} = 1 - c(k+2)^{-\frac{\varepsilon}{2}} \\ \implies u_k &= (u - c_k)_+ > c \left[(k+1)^{-\frac{\varepsilon}{2}} - (k+2)^{-\frac{\varepsilon}{2}} \right], \end{aligned}$$

and

$$\begin{aligned}
& c \left[(k+1)^{-\frac{\varepsilon}{2}} - (k+2)^{-\frac{\varepsilon}{2}} \right] = c(k+1)^{-\frac{\varepsilon}{2}} \left[1 - \left(\frac{k+1}{k+2} \right)^{\frac{\varepsilon}{2}} \right] \\
& = c(k+1)^{-\frac{\varepsilon}{2}} \left(1 - \frac{k+1}{k+2} \right) \frac{\varepsilon}{2} \theta^{\frac{\varepsilon}{2}-1} \quad \text{where } \frac{k+1}{k+2} < \theta < 1 \\
& \geq \frac{\varepsilon}{2} c(k+1)^{-\frac{\varepsilon}{2}} \frac{1}{k+2} \left(\frac{k+1}{k+2} \right)^{\frac{\varepsilon}{2}-1} \geq \frac{\varepsilon}{2} c(k+2)^{-1-\frac{\varepsilon}{2}}.
\end{aligned}$$

Next we use Chebyshev's inequality to obtain

$$(2.8) \quad \left| \left\{ \psi_{k+1} u_k > \frac{\varepsilon}{2} c(k+2)^{-1-\frac{\varepsilon}{2}} \right\} \right| \leq \frac{4c^2}{\varepsilon^2} (k+2)^{2+\varepsilon} \int (\psi_k u_k)^2.$$

Combining (2.4)-(2.8) we obtain

$$\int (\psi_{k+1} u_{k+1})^2 \lesssim (k+1)^{1+\varepsilon} \left(\int_{B_k} u_{k+1}^2 \right) \tilde{\Psi}^{-1} \left(\frac{4c^2}{\varepsilon^2} (k+2)^{2+\varepsilon} \int (\psi_{k+1} u_k)^2 \right),$$

or in terms of the definition of U_k ,

$$(2.9) \quad U_{k+1} \lesssim (k+1)^{1+\varepsilon} U_k \tilde{\Psi}^{-1} \left(\frac{4c^2}{\varepsilon^2} (k+2)^{2+\varepsilon} U_k \right).$$

Now we can determine for which values of parameters α and N the DeGiorgi iteration provides local boundedness of weak subsolutions, i.e. $U_k \rightarrow 0$ as $k \rightarrow \infty$. We need to estimate $\tilde{\Psi}^{-1}$ in order to use (2.9). First note that we can write

$$\Phi(t) = \Phi_N(t) = t(\ln t)^N$$

and therefore

$$\Phi'(t) = (\ln t)^N + tN(\ln t)^{N-1} \frac{1}{t} \geq (\ln t)^N.$$

Denote $s = \Phi'(t)$, we then have

$$\begin{aligned}
s & \geq (\ln t)^N \\
(\Phi')^{-1}(s) & = t \leq e^{s^{\frac{1}{N}}}
\end{aligned}$$

Thus

$$\begin{aligned}
\tilde{\Phi}'(s) & = (\Phi')^{-1}(s) \leq e^{s^{\frac{1}{N}}} \\
\tilde{\Phi}(s) & \leq N s^{1-\frac{1}{N}} e^{s^{\frac{1}{N}}},
\end{aligned}$$

since

$$\begin{aligned}
\frac{d}{ds} \left(N s^{1-\frac{1}{N}} e^{s^{\frac{1}{N}}} \right) & = N s^{1-\frac{1}{N}} e^{s^{\frac{1}{N}}} \frac{1}{N} s^{\frac{1}{N}-1} + N \left(1 - \frac{1}{N} \right) s^{-\frac{1}{N}} e^{s^{\frac{1}{N}}} \\
& = e^{s^{\frac{1}{N}}} + (N-1) s^{-\frac{1}{N}} e^{s^{\frac{1}{N}}} \\
& = e^{s^{\frac{1}{N}}} \left\{ 1 + \frac{N-1}{s^{\frac{1}{N}}} \right\} \geq e^{s^{\frac{1}{N}}} \geq \frac{d}{ds} \tilde{\Phi}(s).
\end{aligned}$$

To estimate $\tilde{\Phi}^{-1}(t)$ we write

$$\begin{aligned}
t & = \tilde{\Phi}(s) \leq N s^{1-\frac{1}{N}} e^{s^{\frac{1}{N}}} \\
\ln t & \leq \ln N + \left(1 - \frac{1}{N} \right) \ln s + s^{\frac{1}{N}} \leq \gamma s^{\frac{1}{N}}, \quad s \gg 1, \\
\tilde{\Phi}^{-1}(t) & = s \geq \left(\frac{1}{\gamma} \ln t \right)^N, \quad t \gg 1.
\end{aligned}$$

Finally,

$$\tilde{\Psi}^{-1}(x) \equiv \frac{1}{\tilde{\Phi}^{-1}(\frac{1}{x})} \leq \frac{1}{\left(\frac{1}{\gamma} \ln \frac{1}{x}\right)^N} = \frac{\gamma^N}{\left(\ln \frac{1}{x}\right)^N}, \quad 0 < x \ll 1.$$

Thus from (2.9) we have

$$U_{k+1} \leq C \frac{(k+1)^{1+\varepsilon} U_k}{\left(\ln \frac{\varepsilon^2}{4c^2} \frac{1}{(k+1)^{(2+\varepsilon)U_k}}\right)^N},$$

and using the notation $b_k \equiv \ln \frac{1}{U_k}$, we have

$$b_{k+1} \geq b_k - (1+\varepsilon) \ln(k+1) + N \ln \left(b_k - (2+\varepsilon) \ln(k+1) - \ln \frac{4c^2}{\varepsilon^2} \right) - \ln C, \quad k \geq 0.$$

We now use induction to show

$$b_k \geq b_0 + k$$

for b_0 taken sufficiently large depending on $N > 1$. Indeed, the claim is trivial for $k = 0$. Assume that we have $b_k \geq b_0 + k$ for some $k \geq 0$. Then

$$b_{k+1} \geq b_0 + k + 1 + N \ln \left(b_0 + k - (2+\varepsilon) \ln(k+1) - \ln \frac{4c^2}{\varepsilon^2} \right) - (2+\varepsilon) \ln(k+1) - 1 - \ln C.$$

Note that for $N > 1 + \varepsilon$ we have

$$N \ln \left(b_0 + k - (2+\varepsilon) \ln(k+1) - \ln \frac{4c^2}{\varepsilon^2} \right) - (1+\varepsilon) \ln(k+1) - 1 - \ln C \rightarrow \infty, \quad \text{as } k \rightarrow \infty$$

and therefore for b_0 sufficiently large depending on N , we obtain

$$N \ln \left(b_0 + k - (2+\varepsilon) \ln(k+1) - \ln \frac{4c^2}{\varepsilon^2} \right) - (1+\varepsilon) \ln(k+1) - 1 - \ln C \geq 0, \quad \forall k \geq 1,$$

which gives

$$b_{k+1} \geq b_0 + k + 1.$$

This proves the induction step and therefore $b_k \rightarrow \infty$ as $k \rightarrow \infty$, or $U_k \rightarrow 0$ as $k \rightarrow \infty$, provided U_0 is sufficiently small. Altogether, this shows that

$$u_\infty = (u-1)_+ = 0 \quad \text{on } B_\infty = B\left(0, \frac{r}{2}\right) = \frac{1}{2}B(0, r),$$

and thus that

$$u \leq 1 \quad \text{on } B_\infty,$$

provided $\|u\|_{L^2(B)}$ is sufficiently small depending on $N > 1 + \varepsilon > 1$, say

$$(2.10) \quad \|u\|_{L^2(B)} < \delta_N.$$

Finally, a rescaling argument gives

$$\|u_+ + \|\phi\|_{X(B)}\|_{L^\infty(\frac{1}{2}B)} \leq \frac{1}{\delta_N} \|u_+ + \|\phi\|_{X(B)}\|_{L^2(B)}.$$

Indeed, set $v_M \equiv \frac{u_+ + \|\phi\|_{X(B)}}{M}$ for $M > 1$. Then using the Cacciopoli inequality (2.2), we obtain

$$\begin{aligned} M^2 \int_B |\nabla_A(\psi v_M)|^2 &= \int_B |\nabla_A(\psi(u_+ + \|\phi\|_X))|^2 \\ &\lesssim \int_B |\nabla_A(\psi u_+)|^2 + \|\phi\|_X^2 \int_B |\nabla_A(\psi)|^2 \\ &\leq C (\|\psi\|_{L^\infty} + \|\nabla_A \psi\|_{L^\infty})^2 \int_B (u_+ + \|\phi\|_X)^2 + \|\phi\|_X^2 \int_B |\nabla_A(\psi)|^2 \\ &\leq C (\|\psi\|_{L^\infty} + \|\nabla_A \psi\|_{L^\infty})^2 M^2 \int_B v^2 \end{aligned}$$

which gives

$$\int_B |\nabla_A(\psi v_M)|^2 \leq C (\|\psi\|_{L^\infty} + \|\nabla_A \psi\|_{L^\infty})^2 \int_B v_M^2$$

uniformly in M . Thus the previous argument shows that

$$v_M \leq 1 \quad \text{on } B_\infty ,$$

provided

$$\int_{B_1} v_M^2 \leq (\delta_N)^2 ,$$

which holds in particular if we choose M so that

$$(\delta_N)^2 = \int_{B_1} v_M^2 = \frac{1}{M^2} \int_{B_1} (u_+ + \|\phi\|_{X(B)})^2 .$$

Thus we obtain

$$\frac{u_+ + \|\phi\|_{X(B)}}{M} = v_M \leq 1 \quad \text{on } B_\infty ,$$

i.e.

$$\|u_+ + \|\phi\|_{X(B)}\|_{L^\infty(\frac{1}{2}B)} \leq M = \frac{1}{\delta_N} \|u_+ + \|\phi\|_{X(B)}\|_{L^2(B)} .$$

This completes the proof of Proposition 12. \square

3. THE ORLICZ BUMP SOBOLEV INEQUALITY IN THE PLANE

This section follows closely the corresponding section in [KoRiSaSh], but with different geometries D_σ and bump functions Φ_N , and we will refer to [KoRiSaSh] throughout our argument here. First, we extend the function Φ_N defined above as $\Phi_N = t(\ln t)^N$ to be convex and non-negative for small values of t , namely

$$(3.1) \quad \Phi_N(t) \equiv \begin{cases} t(\ln t)^N & \text{if } t \geq E = E_N = e^{2N} \\ (\ln E)^N t & \text{if } 0 \leq t \leq E = E_N = e^{2N} \end{cases} ,$$

and the reader can easily verify that Φ_N is submultiplicative for $N \geq 1$ using that for $s, t \geq e^{2N}$,

$$\begin{aligned} st [\ln(st)]^N &= st [\ln s + \ln t]^N \leq s [\ln s]^N t [\ln t]^N \\ \iff \ln s + \ln t &\leq [\ln s] [\ln t] , \end{aligned}$$

and $a + b \leq ab$ if $a, b \geq 2$.

Next, define the positive operator $T_{B(0, r_0)} : L^1(\mu_{r_0}) \rightarrow L^\Phi(\mu_{r_0})$ by

$$T_{B(0, r_0)} g(x) \equiv \int_{B(0, r_0)} K_{B(0, r_0)}(x, y) g(y) dy$$

with kernel $K_{B(0, r_0)}$ defined by

$$K_{B(0, r_0)}(x, y) = \frac{\widehat{d}(x, y)}{|B(x, d(x, y))|} \mathbf{1}_{\Gamma(r_0)}(x, y) ,$$

where $\Gamma(x, r_0)$ is a cusp with peak at x (see [KoRiSaSh] for more detail) and

$$(3.2) \quad \widehat{d}(x, y) \equiv \min \left\{ d(x, y) , \frac{1}{|F'(x_1 + d(x, y))|} \right\} .$$

We consider the following inequality,

$$(3.3) \quad \Phi^{(-1)} \left(\int_{B(0, r_0)} \Phi(T_{B(0, r_0)} g) d\mu_{r_0} \right) \leq C \varphi(r_0) \|g\|_{L^1(\mu_{r_0})} ,$$

which we refer to as the *strong* (Φ, φ) -Sobolev Orlicz bump inequality. This strong inequality (3.3) implies the strong form of the norm inequality

$$(3.4) \quad \|T_{B(0, r_0)} g\|_{L^\Phi(\mu_{r_0})} \leq C \varphi(r_0) \|g\|_{L^1(\mu_{r_0})} ,$$

which in turn implies the norm inequality

$$(3.5) \quad \|w\|_{L^\Phi(\mu_{r_0})} \leq C \varphi(r_0) \|\nabla_A w\|_{L^1(\mu_{r_0})} , \quad w \in W_0^{1,1}(B(0, r_0)) .$$

We begin by proving that the bound (3.3) holds if the following endpoint inequality holds:

$$(3.6) \quad \Phi^{-1} \left(\sup_{y \in B} \int_B \Phi(K(x, y) |B| \alpha) d\mu(x) \right) \leq C \alpha \varphi(r) .$$

for all $\alpha > 0$. Indeed, if (3.6) holds, then with $g = |\nabla_A w|$ and $\alpha = \|g\|_{L^1} = \|\nabla_A w\|_{L^1}$, we have using first the subrepresentation inequality from Chapter 7.1 of [KoRiSaSh],

$$w(x) \leq C \int |\nabla_A w(y)| K_{B(0, r_0)}(x, y) dy, \quad x \in B(0, r), x_1 > 0,$$

$$K_{B(0, r_0)}(x, y) = \frac{\widehat{d}(x, y)}{|B(x, d(x, y))|} \mathbf{1}_{\Gamma(0, r_0)}(x, y) .$$

and then Jensen's inequality applied to the convex function Φ ,

$$\begin{aligned} \int_B \Phi(w) d\mu(x) &\lesssim \int_B \Phi \left(\int_B K(x, y) |B| \|g\|_{L^1(\mu)} \frac{g(y) d\mu(y)}{\|g\|_{L^1(\mu)}} \right) d\mu(x) \\ &\leq \int_B \int_B \Phi(K(x, y) |B| \|g\|_{L^1(\mu)}) \frac{g(y) d\mu(y)}{\|g\|_{L^1(\mu)}} d\mu(x) \\ &\leq \int_B \left\{ \sup_{y \in B} \int_B \Phi(K(x, y) |B| \|g\|_{L^1(\mu)}) d\mu(x) \right\} \frac{g(y) d\mu(y)}{\|g\|_{L^1(\mu)}} \\ &\leq \Phi(C\varphi(r) \|g\|_{L^1(\mu)}) \int_B \frac{g(y) d\mu(y)}{\|g\|_{L^1(\mu)}} = \Phi(C\varphi(r) \|g\|_{L^1(\mu)}), \end{aligned}$$

and so

$$\Phi^{-1} \left(\int_B \Phi(w) d\mu(x) \right) \lesssim C\varphi(r) \|g\|_{L^1(\mu)} .$$

The converse follows from Fatou's lemma, but we will not need this. Note that (3.9) is obtained from (3.3) by replacing $g(y) dy$ with the point mass $|B| \alpha \delta_x(y)$ so that $Tg(x) \rightarrow K(x, y) |B| \alpha$.

Proposition 13. *Assume that for some $\varepsilon > 0$ and $C > 0$ the function*

$$(3.7) \quad \varphi(r) \equiv r^{N+1-\varepsilon} |F'(r)|^N$$

is nondecreasing on $(0, r_0)$. Then:

- (1) *the (Φ, φ) -Sobolev inequality (3.3) holds with geometry F , with φ as in (3.7), and with Φ as in (3.1), $N > 1$,*
- (2) *and if $\varphi_{\max}(r) \equiv \sup_{0 < s < r_0} \varphi(s) < \infty$ is a finite constant function, then the (Φ, φ_{\max}) -Sobolev inequality (3.3) holds with geometry F , with φ as in (3.7), and with Φ as in (3.1), $N > 1$,*
- (3) *and moreover, if we have*

$$(3.8) \quad |F'(r)| \leq C \left(\frac{1}{r} \right)^{1 + \frac{1-\varepsilon}{N}},$$

then the Φ_N -Sobolev inequality (3.3) holds with geometry F and $\varphi_{\max}(r) \equiv C$.

Proof. For Part (1) it suffices to prove the endpoint inequality

$$(3.9) \quad \Phi^{-1} \left(\sup_{y \in B} \int_B \Phi(K(x, y) |B| \alpha) d\mu(x) \right) \leq C \alpha \varphi(r(B)), \quad \alpha > 0.$$

However, since the estimates we use on the kernel $K(x, y)$ are essentially symmetric in x and y , see e.g. the formula (??) for the dual cone Γ^* , we will instead prove the ‘dual’ of (3.9) in which x and y are interchanged:

$$(3.10) \quad \Phi^{-1} \left(\sup_{x \in B} \int_B \Phi(K(x, y) |B| \alpha) d\mu(y) \right) \leq C \alpha \varphi(r(B)), \quad \alpha > 0,$$

for the balls and kernel associated with our geometry F , the Orlicz bump Φ , and the function $\varphi(r)$ satisfying (3.7). Fix parameters $N > 1$ and $t_N > 1$. Now we consider the specific function $\omega(r(B))$ given by

$$\omega(r(B)) = \frac{1}{t_N |F'(r(B))|}.$$

Using the submultiplicativity of Φ we have

$$\begin{aligned} \int_B \Phi(K(x, y)|B|^\alpha) d\mu(y) &= \int_B \Phi\left(\frac{K(x, y)|B|}{\omega(r(B))} \alpha \omega(r(B))\right) d\mu(y) \\ &\leq \Phi(\alpha \omega(r(B))) \int_B \Phi\left(\frac{K(x, y)|B|}{\omega(r(B))}\right) d\mu(y) \end{aligned}$$

and we will now prove

$$(3.11) \quad \int_B \Phi\left(\frac{K(x, y)|B|}{\omega(r(B))}\right) d\mu(y) \leq C_N \varphi(r(B)) |F'(r(B))|,$$

for all small balls B of radius $r(B)$ centered at the origin. Altogether this will give us

$$\int_B \Phi(K(x, y)|B|^\alpha) d\mu(y) \leq C_N \varphi(r(B)) |F'(r(B))| \Phi\left(\frac{\alpha}{t_N |F'(r(B))|}\right).$$

Now we note that $x\Phi(y) = xy \frac{\Phi(y)}{y} \leq xy \frac{\Phi(xy)}{xy} = \Phi(xy)$ for $x \geq 1$ since $\frac{\Phi(t)}{t}$ is monotone increasing. But from (3.7) and assumption (4) on the geometry F we have $\varphi(r) |F'(r)| \gg 1$ and so

$$\int_B \Phi(K(x, y)|B|^\alpha) d\mu(y) \leq \Phi\left(C_N \varphi(r(B)) |F'(r(B))| \alpha \frac{1}{t_N |F'(r(B))|}\right) = \Phi\left(\frac{C_N}{t_N} \alpha \varphi(r(B))\right),$$

which is (3.10) with $C = \frac{C_N}{t_N}$. Thus it remains to prove (3.11).

So we now take $B = B(0, r_0)$ with $r_0 \ll 1$ so that $\omega(r(B)) = \omega(r_0)$. First, recall

$$|B(0, r_0)| \approx \frac{f(r_0)}{|F'(r_0)|^2},$$

and

$$K(x, y) \approx \frac{1}{h_{y_1 - x_1}} \approx \begin{cases} \frac{1}{r f(x_1)}, & 0 < r = y_1 - x_1 < \frac{1}{|F'(x_1)|} \\ \frac{|F'(x_1 + r)|}{f(x_1 + r)}, & 0 < r = y_1 - x_1 \geq \frac{1}{|F'(x_1)|} \end{cases}.$$

Next, write $\Phi(t)$ as

$$(3.12) \quad \Phi(t) = t\Psi(t), \quad \text{for } t > 0,$$

where for $t \geq E$,

$$\begin{aligned} t\Psi(t) &= \Phi(t) = t(\ln t)^N \\ \implies \Psi(t) &= (\ln t)^N, \end{aligned}$$

and for $t < E$,

$$\begin{aligned} t\Psi(t) &= \Phi(t) = t(\ln E)^N \\ \implies \Psi(t) &= (\ln E)^N. \end{aligned}$$

Now temporarily fix $x = (x_1, x_2) \in B_+(0, r_0) \equiv \{x \in B(0, r_0) : x_1 > 0\}$. We then have for $-x_1 < a < b < r_0 - x_1$ that

$$\begin{aligned} \mathcal{I}_{a,b}(x) &\equiv \int_{\{y \in B_+(0, r_0) : a \leq y_1 - x_1 \leq b\}} \Phi\left(K_{B(0, r_0)}(x, y) \frac{|B(0, r_0)|}{\omega(r_0)}\right) \frac{dy}{|B(0, r_0)|} \\ &= \int_{a+x_1}^{b+x_1} \left\{ \int_{x_2-h_{y_1-x_1}}^{x_2+h_{y_1-x_1}} \Phi\left(\frac{1}{h_{y_1-x_1}} |B(0, r_0)| \frac{|B(0, r_0)|}{\omega(r_0)}\right) dy_2 \right\} \frac{dy_1}{|B(0, r_0)|} \\ &= \int_{a+x_1}^{b+x_1} 2h_{y_1-x_1} \Phi\left(\frac{1}{h_{y_1-x_1}} \frac{|B(0, r_0)|}{\omega(r_0)}\right) \frac{dy_1}{|B(0, r_0)|} \\ &= \int_{a+x_1}^{b+x_1} 2h_{y_1-x_1} \left(\frac{1}{h_{y_1-x_1}} \frac{|B(0, r_0)|}{\omega(r_0)}\right) \Psi\left(\frac{1}{h_{y_1-x_1}} \frac{|B(0, r_0)|}{\omega(r_0)}\right) \frac{dy_1}{|B(0, r_0)|} \end{aligned}$$

which simplifies to

$$\begin{aligned}\mathcal{I}_{a,b}(x) &= \frac{2}{\omega(r_0)} \int_{a+x_1}^{b+x_1} \Psi\left(\frac{1}{h_{y_1-x_1}} \frac{|B(0, r_0)|}{\omega(r_0)}\right) dy_1 \\ &= \frac{2}{\omega(r_0)} \int_a^b \Psi\left(\frac{1}{h_r} \frac{|B(0, r_0)|}{\omega(r_0)}\right) dr.\end{aligned}$$

Thus we have

$$\begin{aligned}& \int_{B_+(0, r_0)} \Phi\left(K_{B(0, r_0)}(x, y) \frac{|B(0, r_0)|}{\omega(r_0)}\right) \frac{dy}{|B(0, r_0)|} \\ &= \mathcal{I}_{-x_1, r_0-x_1}(x) \\ &= \frac{2}{\omega(r_0)} \int_{-x_1}^{r_0-x_1} \Psi\left(\frac{1}{h_r} \frac{|B(0, r_0)|}{\omega(r_0)}\right) dr.\end{aligned}$$

To prove (3.11) it suffices to obtain the following estimate for the integral \mathcal{I}_{0, r_0-x_1} , since the complementary integral $\mathcal{I}_{-x_1, 0}$ can be handled similarly to obtain the same estimate:

$$(3.13) \quad \mathcal{I}_{0, r_0-x_1} = \frac{1}{\omega(r_0)} \int_0^{r_0-x_1} \Psi\left(\frac{|B(0, r_0)|}{h_r \omega(r_0)}\right) dr \leq C_N \varphi(r_0) |F'(r_0)|,$$

where C_0 is a sufficiently large positive constant.

To prove this we divide the interval $(0, r_0 - x_1)$ of integration in r into three regions:

- (1): the small region \mathcal{S} where $\frac{|B(0, r_0)|}{h_r \omega(r_0)} \leq E$,
- (2): the big region \mathcal{R}_1 that is disjoint from \mathcal{S} and where $r = y_1 - x_1 < \frac{1}{|F'(x_1)|}$ and
- (3): the big region \mathcal{R}_2 that is disjoint from \mathcal{S} and where $r = y_1 - x_1 \geq \frac{1}{|F'(x_1)|}$.

In the small region \mathcal{S} we use that Φ is linear on $[0, E]$ to obtain that the integral in the right hand side of (3.13), when restricted to those $r \in (0, r_0 - x_1)$ for which $\frac{|B(0, r_0)|}{h_r \omega(r_0)} \leq E$, is equal to

$$\begin{aligned}& \frac{1}{\omega(r_0)} \int_0^{r_0-x_1} \Psi\left(\frac{|B(0, r_0)|}{h_r \omega(r_0)}\right) dr \\ &= \frac{1}{\omega(r_0)} \int_0^{r_0-x_1} \frac{\Phi(E)}{E} dr = \frac{1}{\omega(r_0)} (\ln E)^N (r_0 - x_1) \\ &\leq C t_N r_0 |F'(r_0)| \leq C_N \varphi(r_0) |F'(r_0)|,\end{aligned}$$

since $\omega(r_0) = \frac{1}{t_N |F'(r_0)|}$, and for the last inequality we used $r_0 \leq \varphi(r_0)$ which follows from (3.7) and assumption (4) on the geometry that $r_0 |F'(r_0)| \geq C$.

We now turn to the first big region \mathcal{R}_1 where we have $h_{y_1-x_1} \approx r f(x_1)$. The condition that \mathcal{R}_1 is disjoint from \mathcal{S} gives

$$\begin{aligned}\frac{|B(0, r_0)|}{r f(x_1) \omega(r_0)} &> E, \quad \text{i.e. } r < \frac{A}{E}; \\ \text{where } A &= A(x_1) \equiv \frac{|B(0, r_0)|}{f(x_1) \omega(r_0)},\end{aligned}$$

and so

$$\begin{aligned}& \int_{\mathcal{R}_1} \Phi\left(K_{B(0, r_0)}(x, y) \frac{|B(0, r_0)|}{\omega(r_0)}\right) \frac{dy}{|B(0, r_0)|} \\ &= \mathcal{I}_{0, \min\left\{\frac{A}{E}, \frac{1}{|F'(x_1)|}\right\}}(x) \\ &= \frac{1}{\omega(r_0)} \int_0^{\min\left\{\frac{A}{E}, \frac{1}{|F'(x_1)|}\right\}} \Psi\left(\frac{|B(0, r_0)|}{h_r \omega(r_0)}\right) dr \\ &= \frac{1}{\omega(r_0)} \int_0^{\min\left\{\frac{A}{E}, \frac{1}{|F'(x_1)|}\right\}} \Psi\left(\frac{A}{r}\right) dr.\end{aligned}$$

We will now show that

$$(3.14) \quad \frac{1}{\omega(r_0)} \int_0^{\min\left\{\frac{A}{E}, \frac{1}{|F'(x_1)|}\right\}} \Psi\left(\frac{A}{r}\right) dr \lesssim C_N \varphi(r_0) |F'(r_0)|,$$

where we recall

$$\begin{aligned} A &= A(x_1) \equiv \frac{f(r_0)}{f(x_1) |F'(r_0)|^2 \omega(r_0)} = \frac{c}{f(x_1)} \\ \text{and } c &= c(r_0) \equiv \frac{f(r_0)}{\omega(r_0) |F'(r_0)|^2} = \frac{t_N f(r_0)}{|F'(r_0)|}. \end{aligned}$$

Now if $\frac{A}{E} \leq \frac{1}{|F'(x_1)|}$, then

$$\begin{aligned} & \frac{1}{\omega(r_0)} \int_0^{\min\left\{\frac{A}{E}, \frac{1}{|F'(x_1)|}\right\}} \Psi\left(\frac{A}{r}\right) dr \\ &= \frac{1}{\omega(r_0)} \int_0^{\frac{A}{E}} \Psi\left(\frac{A}{r}\right) dr = \frac{1}{\omega(r_0)} A \int_E^\infty \Psi(t) \frac{dt}{t^2} = \frac{1}{\omega(r_0)} A \int_E^\infty (\ln t)^N \frac{dt}{t^2} \\ &\leq \frac{C_N A}{\omega(r_0)} \leq \frac{C_N |F'(r_0)|}{|F'(x_1)|} \leq C_N, \end{aligned}$$

which proves (3.13) if $\frac{A}{E} \leq \frac{1}{|F'(x_1)|}$ since $\varphi(r_0) |F'(r_0)| \geq 1$.

So we now suppose that $\frac{A}{E} > \frac{1}{|F'(x_1)|}$. Making a change of variables

$$R = \frac{A}{r} = \frac{A(x_1)}{r},$$

we obtain

$$\frac{1}{\omega(r_0)} \int_0^{\frac{1}{|F'(x_1)|}} \Psi\left(\frac{A}{r}\right) dr = \frac{1}{\omega(r_0)} A \int_{A|F'(x_1)|}^\infty \frac{(\ln R)^N}{R^2} dR.$$

Integrating by parts gives

$$\begin{aligned} \int_{A|F'(x_1)|}^\infty \frac{(\ln R)^N}{R^2} dR &= \int_{A|F'(x_1)|}^\infty R (\ln R)^N \left(-\frac{1}{2R^2}\right)' dR \\ &= -\frac{R (\ln R)^N}{2R^2} \Big|_{A|F'(x_1)|}^\infty + \int_{A|F'(x_1)|}^\infty \left(R (\ln R)^N\right)' \frac{1}{2R^2} dR \\ &\leq \frac{(\ln(A|F'(x_1)|))^N}{2A|F'(x_1)|} + \int_{A|F'(x_1)|}^\infty \frac{(\ln R)^N}{R^2} \left(1 + \frac{N}{\ln R}\right) dR \\ &\leq \frac{(\ln(A|F'(x_1)|))^N}{2A|F'(x_1)|} + \frac{1 + \frac{N}{\ln E}}{2} \int_{A|F'(x_1)|}^\infty \frac{(\ln R)^N}{R^2} dR, \end{aligned}$$

Using the definition of $E = e^{2N}$ we get the second term on the right is equal to

$$\frac{3}{4} \int_{A|F'(x_1)|}^\infty \frac{(\ln R)^N}{R^2} dR$$

and therefore we obtain

$$\int_{A|F'(x_1)|}^\infty \frac{(\ln R)^N}{R^2} dR \leq C \frac{(\ln(A|F'(x_1)|))^N}{A|F'(x_1)|}$$

and therefore

$$\begin{aligned}
\mathcal{I}_{0, \frac{1}{|F'(x_1)|}}(x) &= \frac{1}{\omega(r_0)} A \int_{A|F'(x_1)|}^{\infty} \frac{(\ln R)^N}{R^2} dR \\
&\lesssim \frac{1}{\omega(r_0) |F'(x_1)|} (\ln(A|F'(x_1)|))^N \\
&= t_m |F'(r_0)| \frac{1}{|F'(x_1)|} \left(\ln \left(c(r_0) \frac{|F'(x_1)|}{f(x_1)} \right) \right)^N; \\
c(r_0) &= f(x_1) A(x_1) = \frac{f(r_0)}{\omega(r_0) |F'(r_0)|^2} = \frac{t_m f(r_0)}{|F'(r_0)|},
\end{aligned}$$

where we recall that we have assumed the condition

$$(3.15) \quad A(x_1) |F'(x_1)| = \frac{f(r_0)}{f(x_1) |F'(r_0)|^2 \omega(r_0)} |F'(x_1)| = c(r_0) \frac{|F'(x_1)|}{f(x_1)} \geq E.$$

We thus need to show

$$\frac{1}{|F'(x_1)|} \left(\ln \left(c(r_0) \frac{|F'(x_1)|}{f(x_1)} \right) \right)^N = \frac{1}{|F'(x_1)|} (\ln c(r_0) + \ln |F'(x_1)| + F(x_1))^N \leq \varphi(r_0).$$

Define

$$(3.16) \quad \mathcal{F}(x_1) \equiv \frac{1}{|F'(x_1)|} (\ln c(r_0) + \ln |F'(x_1)| + F(x_1))^N.$$

and note that it is sufficient to show

$$(3.17) \quad \sup_{x_1 \in (0, r_0)} \mathcal{F}(x_1) \leq \varphi(r_0)$$

which will prove the estimate (3.13) for the region \mathcal{R}_1 . We will look for the maximum of $\mathcal{F}(x_1)$ on $(0, r_0)$. Differentiating $\mathcal{F}(x_1)$ and setting the derivative equal to zero, we obtain the following implicit expression for x_1^* maximizing $\mathcal{F}(x_1)$:

$$\begin{aligned}
&\frac{F''(x_1^*)}{|F'(x_1^*)|^2} (\ln c(r_0) + \ln |F'(x_1^*)| + F(x_1^*))^N \\
&+ \frac{N}{|F'(x_1^*)|} (\ln c(r_0) + \ln |F'(x_1^*)| + F(x_1^*))^{N-1} \left(-\frac{F''(x_1^*)}{|F'(x_1^*)|} - |F'(x_1^*)| \right) = 0,
\end{aligned}$$

which gives

$$(\ln c(r_0) + \ln |F'(x_1^*)| + F(x_1^*)) = N \left(1 + \frac{|F'(x_1^*)|^2}{F''(x_1^*)} \right).$$

Substituting into (3.16) gives

$$\mathcal{F}(x_1^*) = \frac{(N)^N}{|F'(x_1^*)|} \left(1 + \frac{|F'(x_1^*)|^2}{F''(x_1^*)} \right)^N \leq C_N (x_1^*)^N |F'(x_1^*)|^{N-1}.$$

where we used properties (5) and (4) on the geometry F . Using the monotonicity assumption (3.7) and property (4) of the geometry we obtain

$$\mathcal{F}(x_1) \leq C_N \varphi(x_1^*) \leq C_N \varphi(r_0).$$

For the second big region \mathcal{R}_2 we have

$$\frac{1}{h_{y_1-x_1}} \approx \frac{|F'(x_1+r)|}{f(x_1+r)},$$

and the integral to be estimated becomes

$$I_{\mathcal{R}_2} \equiv \frac{1}{\omega(r_0)} \int_{x_1 + \frac{1}{|F'(x_1)|}}^{r_0} \left(\ln \left(\frac{f(r_0) |F'(y_1)|}{f(y_1) |F'(r_0)|^2 \omega(r_0)} \right) \right)^N dy_1.$$

We still have the condition (3.15) for this integral, i.e.

$$(3.18) \quad A(y_1) |F'(y_1)| = \frac{f(r_0)}{f(y_1) |F'(r_0)|^2 \omega(r_0)} |F'(y_1)| \geq E.$$

Again, we would like to estimate the above integral by $C_N \varphi(r_0) |F'(r_0)|$. Write

$$\int_{x_1 + \frac{1}{|F'(x_1)|}}^{r_0} \left(\ln \left(c(r_0) \frac{|F'(y_1)|}{f(y_1)} \right) \right)^N dy_1 = \int_{y_1 + \frac{1}{|F'(y_1)|}}^{r_0} y_1^{1-\varepsilon} \left(\ln \left(c(r_0) \frac{|F'(y_1)|}{f(y_1)} \right) \right)^N \frac{dy_1}{y_1^{1-\varepsilon}}$$

where

$$c(r_0) = \frac{t_m f(r_0)}{|F'(r_0)|}$$

as before, and define

$$(3.19) \quad \mathcal{G}(y_1) \equiv y_1^{1-\varepsilon} (\ln c(r_0) + \ln |F'(y_1)| + F(y_1))^N.$$

Again, we maximize the function $\mathcal{G}(y_1)$ in a similar way to obtain

$$\mathcal{G}(y_1^*) \leq C(y_1^*)^{N+1-\varepsilon} \left(\frac{F''(y_1^*)}{|F'(y_1^*)|} + |F'(y_1^*)| \right)^N \leq C_N (y_1^*)^{N+1-\varepsilon} |F'(y_1^*)|^N \leq C_N \varphi(r_0).$$

This concludes the proof of part (1).

Part (2) follows from a simple inspection of the arguments above, and noting that appropriate upper bounds can always be taken to be the constant function φ_{\max} . Part (3) is then an immediate consequence, and this completes the proof of Proposition 13. \square

3.1. Two dimensional local boundedness.

Corollary 14. *The Orlicz-Sobolev bump inequality with $\Phi = \Phi_N$ holds for the geometry D_σ if $0 < \sigma < \frac{1}{N}$.*

Proof. It is trivial to check that (3.7) holds for D_σ provided $0 < \sigma < \frac{1}{N}$ using

$$f(x) = e^{-\frac{1}{x^\sigma}}, \quad F(x) = \frac{1}{x^\sigma}, \quad |F'(x)| = \frac{\sigma}{x^{\sigma+1}}, \quad F''(x) = \frac{\sigma(\sigma+1)}{x^{\sigma+2}}.$$

\square

Corollary 15. *The Orlicz-Sobolev bump inequality with $\Phi = \Phi_N$ holds for the geometry F provided $F \leq D_\sigma$ for some $0 < \sigma < \frac{1}{N}$.*

Proof. It follows from Proposition 13 that it suffices to show

$$|F'(r)| \leq C \left(\frac{1}{r} \right)^{1+\frac{1-\varepsilon}{N}}.$$

We claim

$$(3.20) \quad x|F'(x)| \leq F(x), \text{ for } x \in (0, r_0).$$

Indeed, we have

$$\frac{1}{F(x)} = \int_0^x \left(\frac{1}{F(y)} \right)' dy = - \int_0^x \frac{F'(y)}{F(y)^2} dy,$$

where we used structure condition (1) that $\lim_{x \rightarrow 0^+} F(x) = 0$. Now using structure condition (2) we get that $\frac{F'(y)}{F(y)^2}$ is an increasing function of y on $(0, r_0)$, and therefore

$$\frac{1}{F(x)} = - \int_0^x \frac{F'(y)}{F(y)^2} dy \geq - \frac{F'(x)}{F(x)^2} x,$$

which gives

$$F(x) \geq -x F'(x),$$

which concludes the proof of (3.20) since $F'(x) < 0$ by structure condition (2).

We therefore get

$$|F'(r)| \leq \frac{F(r)}{r} \leq \frac{D_\sigma(r)}{r} = \frac{1}{r^{\sigma+1}}$$

Thus by (3.8) of Proposition 13 the Orlicz-Sobolev inequality for geometry F holds provided $1 + \sigma \leq \frac{1-\varepsilon}{N}$ for some $\varepsilon > 0$. This concludes the proof. \square

Corollary 16. *Weak subsolutions to $\mathcal{L}u = \phi$ with A -admissible ϕ are locally bounded for the geometry F provided $F \leq D_\sigma$ for some $0 < \sigma < 1$.*

Proof. Above we showed that weak solutions are locally bounded provided that the Sobolev inequality with Φ_N bump (3.1) holds for some $N > 1$. Combining this with Corollary 15 and Proposition 12, we obtain the desired result. \square

3.2. Two dimensional maximum principle. In this subsection we prove a maximum principle for weak subsolutions under a very weak assumption on the geometry F . Namely, we will assume that $f(x) \neq 0$ if $x \neq 0$, and that F satisfies the five structural conditions in Definition 7. We need the following Sobolev inequality

$$(3.21) \quad \|w\|_{L^\Phi(\Omega)} \leq \varphi(\Omega) \|\nabla_A w\|_{L^1(\Omega)}, \quad w \in \left(W_A^{1,2}\right)_0(\Omega)$$

where the assumptions on the bump Φ are very mild. Namely, we will require

$$\frac{\Phi(t)}{t} \rightarrow \infty, \quad \text{as } t \rightarrow \infty.$$

As before we write $\Phi(t) = t^{1+\psi(t)}$ and note from above $t^{\psi(t)} \rightarrow \infty$ as $t \rightarrow \infty$.

We now prove the following global boundedness result.

Theorem 17. *Let Ω be a domain in the plane and let F be any geometry satisfying the structure conditions in Definition 7. Assume that u is a weak subsolution to $\mathcal{L}u = \phi$ in Ω with A -admissible ϕ , and that u is bounded on the boundary $\partial\Omega$. Then the following maximum principle holds,*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C \|\phi\|_{X(\Omega)},$$

and in particular u is globally bounded.

Proof. First we assume in addition that the general Orlicz-Sobolev inequality (3.21) holds. By proceeding as in Section 2 we can see that $u_k \equiv 0$ for all k on $\partial\Omega$, so we can formally take $\psi_k \equiv 1$ for all k . This leads to inequality (2.9) being replaced by the following inequality

$$U_{k+1} \lesssim C U_k \tilde{\Psi}^{-1}(k^5 U_k)$$

with a constant C independent of k , where $U_k = \int_{\Omega} |u_k|^2$. Recalling the notation $b_k \equiv -\ln U_k$, we have

$$b_{k+1} \geq b_k - \ln C - \ln \tilde{\Psi}^{-1}(e^{-b_k+5 \ln k})$$

Now denote

$$h(t) \equiv -\ln \tilde{\Psi}^{-1}(e^{-t})$$

and assume the following (see Lemma 18 below)

- (1) $h > 0$,
- (2) h is increasing,
- (3) $h(t) \rightarrow \infty$ as $t \rightarrow \infty$.

With this notation

$$b_{k+1} \geq b_k - \ln C + h(b_k - 5 \ln k).$$

We now prove by induction that $b_k \geq b_0 + k$ for b_0 large enough depending only on h . Indeed, the claim is obvious for $k = 0$. Assume the claim is true for k . Then

$$b_{k+1} \geq b_0 + k - \ln C + h(b_0 + k - 5 \ln k) = b_0 + k + 1 + (h(b_0 + k - 5 \ln k) - \ln C - 1).$$

By our assumptions on h we have

$$h(b_0 + k - 5 \ln k) - \ln C - 1 \rightarrow \infty, \quad \text{as } k \rightarrow \infty,$$

and it is an increasing function of b_0 . Therefore, by choosing b_0 large enough depending on h we can guarantee

$$h(b_0 + k - 5 \ln k) - \ln C - 1 > 0, \quad \text{for all } k,$$

and thus

$$b_{k+1} \geq b_0 + k + 1.$$

This concludes the proof of the induction step, and therefore we get

$$\begin{aligned} b_k &\rightarrow \infty, \quad \text{as } k \rightarrow \infty, \\ U_k &\rightarrow 0, \quad \text{as } k \rightarrow \infty, \end{aligned}$$

and the maximum principle and global boundedness follow.

Now we give a proof of the Sobolev inequality (3.21). We first proceed in the same way we did in Section 3 to get the following sufficient condition for Φ -Sobolev inequality

$$(3.22) \quad \int_{B(0,r_0)} \Phi \left(\frac{K_{B(0,r_0)}(x,y) |B(0,r_0)|}{\varphi(r_0)} \right) \frac{dy}{|B(0,r_0)|} \leq 1.$$

Now we write

$$(3.23) \quad \int_{B(0,r_0)} \Phi \left(\frac{K_{B(0,r_0)}(x,y) |B(0,r_0)|}{\varphi(r_0)} \right) \frac{dy}{|B(0,r_0)|} = \int_0^{r_0} \frac{1}{\varphi(r_0)} \frac{\Phi \left(\frac{|B(0,r_0)|}{\varphi(r_0)h_r} \right)}{\frac{|B(0,r_0)|}{\varphi(r_0)hr}} dr,$$

and note the following. The function $\widehat{\Phi}(t) \equiv \frac{\Phi(t)}{t}$ satisfies

$$(3.24) \quad \lim_{t \rightarrow \infty} \widehat{\Phi}(t) = \infty,$$

and moreover, Φ can be chosen so that $\widehat{\Phi}$ is a monotone increasing function of its argument. Next, by definition we have that $h_r \rightarrow 0$ as $r \rightarrow 0$. Therefore,

$$\widehat{\Phi} \left(\frac{|B(0,r_0)|}{\varphi(r_0)h_r} \right) \rightarrow \infty, \quad \text{as } r \rightarrow 0,$$

and $\widehat{\Phi} \left(\frac{|B(0,r_0)|}{\varphi(r_0)h_r} \right)$ is a decreasing function of $\varphi(r_0)$. Thus we can choose Φ to be convex and satisfy (3.24) so that

$$\frac{1}{\varphi(r_0)} \frac{\Phi \left(\frac{|B(0,r_0)|}{\varphi(r_0)h_r} \right)}{\frac{|B(0,r_0)|}{\varphi(r_0)hr}} \leq \frac{C_{r_0}}{r^{1/2}},$$

and the integral in (3.23) converges. Moreover, choosing $\varphi(r_0)$ sufficiently large, we can satisfy (3.22). \square

Lemma 18. *Let $\Phi(t)$ be a Young function that satisfies*

$$(3.25) \quad \frac{\Phi(t)}{t} \rightarrow \infty, \quad \text{as } t \rightarrow \infty,$$

and let $\tilde{\Phi}(t)$ its convex conjugate. Define

$$\begin{aligned} \tilde{\Psi}^{-1}(s) &\equiv \frac{1}{\tilde{\Phi}^{-1}\left(\frac{1}{s}\right)}, \\ h(t) &\equiv -\ln \tilde{\Psi}^{-1}(e^{-t}). \end{aligned}$$

Then we have

$$(3.26) \quad h(t) \rightarrow \infty, \quad \text{as } t \rightarrow \infty.$$

Proof. First note that from the definitions of h and $\tilde{\Psi}$, (3.26) is equivalent to requiring

$$\tilde{\Psi}^{-1}(s) \rightarrow 0, \quad \text{as } s \rightarrow 0 \iff \tilde{\Phi}^{-1}(t) \rightarrow \infty, \quad \text{as } t \rightarrow \infty.$$

Now recall

$$\tilde{\Phi}'(t) = (\Phi')^{-1}(t)$$

and note that

$$(\Phi')^{-1}(t) \rightarrow \infty, \quad \text{as } t \rightarrow \infty \iff \Phi'(t) \rightarrow \infty, \quad \text{as } t \rightarrow \infty.$$

Thus to show (3.26) it is sufficient to show $\Phi'(t) \rightarrow \infty$ as $t \rightarrow \infty$. From a simple integration by parts we get

$$\Phi'(t) = \frac{\Phi(t)}{t} + \frac{1}{t} \int_0^t x \Phi''(x) dx.$$

Thus from convexity of Φ and condition (3.25), we obtain the required estimate. \square

4. THE ORLICZ BUMP SOBOLEV INEQUALITY IN HIGHER DIMENSIONS

Recall that in the two dimensional case, we had

$$|B_{2D}(x, d(x, y))| \approx h_{x,y} \widehat{d}(x, y) \approx h_{x,y} \min \left\{ d(x, y), \frac{1}{|F'(x_1 + d(x, y))|} \right\}.$$

In the three dimensional case, the quantities $h_{x,y}$ and $\widehat{d}(x, y)$ remain formally the same (see Chapter 10 of [KoRiSaSh]) and we can write a typical geodesic in the form

$$\begin{cases} x_2 = C_2 \pm k \int_0^{x_1} \frac{\lambda}{\sqrt{\lambda^2 - [f(u)]^2}} du \\ x_3 = C_3 \pm \int_0^{x_1} \frac{[f(u)]^2}{\sqrt{\lambda^2 - [f(u)]^2}} du \end{cases},$$

so that a metric ball centered at $y = (y_1, y_2, y_3)$ with radius $r > 0$ is given by

$$B(y, r) \equiv \left\{ (x_1, x_2, x_3) : (x_1, x_3) \in B_{2D} \left((y_1, y_3), \sqrt{r^2 - |x_2 - y_2|^2} \right) \right\},$$

where $B_{2D}(a, s)$ denotes the 2-dimensional control ball centered at a in the plane parallel to the x_1, x_3 -plane with radius s that was associated with f above (see Corollaries 107 and 108 in [KoRiSaSh] and the subsequent paragraph).

In dimension $n \geq 4$, the same arguments show that a typical geodesic has the form

$$\begin{cases} \mathbf{x}_2 = \mathbf{C}_2 \pm \mathbf{k} \int_0^{x_1} \frac{\lambda}{\sqrt{\lambda^2 - [f(u)]^2}} du \\ x_3 = C_3 \pm \int_0^{x_1} \frac{[f(u)]^2}{\sqrt{\lambda^2 - [f(u)]^2}} du \end{cases},$$

where $\mathbf{x}_2, \mathbf{C}_2, \mathbf{k} \in \mathbb{R}^{n-2}$ are now $(n-2)$ -dimensional vectors, so that a metric ball centered at

$$y = (y_1, \mathbf{y}_2, y_3) \in \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R} = \mathbb{R}^n,$$

with radius $r > 0$ is given by

$$B(y, r) \equiv \left\{ (x_1, x_2, x_3) : (x_1, x_3) \in B_{2D} \left((y_1, y_3), \sqrt{r^2 - |\mathbf{x}_2 - \mathbf{y}_2|^2} \right) \right\},$$

where $B_{2D}(a, s)$ denotes the 2-dimensional control ball centered at a in the plane parallel to the x_1, x_3 -plane with radius s that was associated with f above.

However, we then claimed in Lemma 109 in Chapter 10 of [KoRiSaSh] that in three dimensions we have the estimate

$$|B(x, d(x, y))| \approx h_{x,y} \widehat{d}(x, y)^2,$$

and hence also the estimate,

$$\begin{aligned} K_B(x, y) &= \frac{\widehat{d}(x, y)}{|B(x, d(x, y))|} \mathbf{1}_{\Gamma(x, r_0)}(y) \\ &\approx \frac{1}{\widehat{d}(x, y) h_{y_1 - x_1}} \mathbf{1}_{\Gamma(x, r_0)}(y) \approx \begin{cases} \frac{1}{r^2 f(x_1)} \mathbf{1}_{\Gamma(x, r_0)}(y), & 0 < r = y_1 - x_1 < \frac{1}{|F'(x_1)|} \\ \frac{|F'(x_1 + r)|^2}{f(x_1 + r)} \mathbf{1}_{\Gamma(x, r_0)}(y), & 0 < r = y_1 - x_1 \geq \frac{1}{|F'(x_1)|} \end{cases}, \end{aligned}$$

where $\Gamma(x, r_0)$ is defined in Subsection 1.2 of Chapter 10 of [KoRiSaSh] using the sequence $\{r_k\}_{k=0}^\infty$ given in (7.1) of [KoRiSaSh]. Unfortunately, these two estimates are off by a factor of $\sqrt{d(x, y) |F'(x_1 + d(x, y))|}$ in the case that $r \geq \frac{2}{|F'(x_1)|}$. We now correct this error in Lemma 109 of [KoRiSaSh] by replacing it with the following lemma instead, which also includes the general n -dimensional case.

Lemma 19. *The Lebesgue measure of the three dimensional ball $B_{3D}(x, r)$ satisfies*

$$|B_{3D}(x, r)| \approx \begin{cases} r^3 f(x_1) & \text{if } r \leq \frac{2}{|F'(x_1)|} \\ \frac{f(x_1 + r)}{|F'(x_1 + r)|^3} \sqrt{r |F'(x_1 + r)|} & \text{if } r \geq \frac{2}{|F'(x_1)|} \end{cases},$$

and that of the n -dimensional ball $B_{nD}(x, r)$ satisfies

$$|B_{nD}(x, r)| \approx \begin{cases} r^n f(x_1) & \text{if } r \leq \frac{2}{|F'(x_1)|} \\ \frac{f(x_1+r)}{|F'(x_1+r)|^n} (r |F'(x_1+r)|)^{\frac{n}{2}-1} & \text{if } r \geq \frac{2}{|F'(x_1)|} \end{cases},$$

Proof. We estimate the measure $|B(x, r)|$ of an n -dimensional ball $B(x, r) = B_{nD}(x, r)$, where $x = (x_1, \mathbf{x}_2, x_3) \in \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R} = \mathbb{R}^n$, and where we use boldface font for \mathbf{x}_2 to emphasize that it belongs to \mathbb{R}^{n-2} as opposed to \mathbb{R} . We consider two cases, where we may assume by symmetry that $\mathbf{x}_2 = \mathbf{0}$ and $x_3 = 0$.

Case $r < \frac{2}{|F'(x_1)|}$: In this case we have $\sqrt{r^2 - |\mathbf{y}_2|^2} \leq r < \frac{2}{|F'(x_1)|}$ and

$$\left| B_{2D} \left((x_1, \mathbf{0}, 0), \sqrt{r^2 - |\mathbf{y}_2|^2} \right) \right| \approx (r^2 - |\mathbf{y}_2|^2) f(x_1),$$

where for $r < \frac{1}{|F'(x_1)|}$ we appeal to the second assertion in Proposition 81 of [KoRiSaSh], while for $\frac{1}{|F'(x_1)|} \leq r < \frac{2}{|F'(x_1)|}$ we appeal to the first assertion in Proposition 81 of [KoRiSaSh] and use the estimates for f and $|F'|$ in (2) of Lemma 69 in [KoRiSaSh]. With $A(a, b) \equiv \{\mathbf{y}_2 \in \mathbb{R}^{n-2} : a \leq |\mathbf{y}_2| \leq b\}$ denoting the annulus centered at the origin in \mathbb{R}^{n-2} with radii $a < b$, the above gives

$$|B(x, r)| = \int_{A(0, r)} \left| B_{2D} \left((x_1, \mathbf{0}, 0), \sqrt{r^2 - |\mathbf{y}_2|^2} \right) \right| d\mathbf{y}_2 \approx \int_{A(0, r)} (r^2 - |\mathbf{y}_2|^2) f(x_1) d\mathbf{y}_2 \approx r^n f(x_1).$$

Case $r \geq \frac{2}{|F'(x_1)|}$: In this case the integral in $|\mathbf{y}_2| \leq r$ is divided into two regions.

Region 1: $0 < \sqrt{r^2 - |\mathbf{y}_2|^2} \leq \frac{1}{|F'(x_1)|}$. In this region we have $\sqrt{r^2 - \frac{1}{|F'(x_1)|^2}} \leq |\mathbf{y}_2| \leq r$ and

$$\left| B_{2D} \left((x_1, \mathbf{0}, 0), \sqrt{r^2 - |\mathbf{y}_2|^2} \right) \right| \approx (r^2 - |\mathbf{y}_2|^2) f(x_1).$$

Thus we obtain

$$\begin{aligned} \int_{A\left(\sqrt{r^2 - \frac{1}{|F'(x_1)|^2}}, r\right)} \left| B_{2D} \left((x_1, \mathbf{0}, 0), \sqrt{r^2 - |\mathbf{y}_2|^2} \right) \right| d\mathbf{y}_2 &\approx \int_{A\left(\sqrt{r^2 - \frac{1}{|F'(x_1)|^2}}, r\right)} (r^2 - |\mathbf{y}_2|^2) f(x_1) d\mathbf{y}_2 \\ &= f(x_1) \left[\frac{c_n r^2}{n-2} \left(r^{n-2} - \left(r^2 - \frac{1}{|F'(x_1)|^2} \right)^{\frac{n}{2}-1} \right) - \frac{c_n}{n} \left(r^n - \left(r^2 - \frac{1}{|F'(x_1)|^2} \right)^{\frac{n}{2}} \right) \right] \\ &\approx \frac{r^{n-2} f(x_1)}{|F'(x_1)|^2}, \end{aligned}$$

where we have used the estimate

$$\begin{aligned} r^\alpha - \left(r^2 - \frac{1}{|F'(x_1)|^2} \right)^{\frac{\alpha}{2}} &= r^\alpha \left\{ 1 - \left(1 - \frac{1}{r^2 |F'(x_1)|^2} \right)^{\frac{\alpha}{2}} \right\} \\ &\approx r^\alpha \left\{ 1 - \left(1 - \frac{\alpha}{2} \frac{1}{r^2 |F'(x_1)|^2} \right) \right\} \approx \frac{r^{\alpha-2}}{|F'(x_1)|^2}, \end{aligned}$$

since $\frac{1}{r^2 |F'(x_1)|^2} \leq \frac{1}{4}$ in the present case $r \geq \frac{2}{|F'(x_1)|}$.

Region 2: $\frac{1}{|F'(x_1)|} \leq \sqrt{r^2 - |\mathbf{y}_2|^2} \leq r$. In this region we have $0 \leq |\mathbf{y}_2| \leq \sqrt{r^2 - \frac{1}{|F'(x_1)|^2}}$ and

$$\left| B_{2D} \left((x_1, \mathbf{0}, 0), \sqrt{r^2 - |\mathbf{y}_2|^2} \right) \right| \approx \frac{f \left(x_1 + \sqrt{r^2 - |\mathbf{y}_2|^2} \right)}{\left| F' \left(x_1 + \sqrt{r^2 - |\mathbf{y}_2|^2} \right) \right|^2}.$$

We now claim that

$$(4.1) \quad \int_{A\left(0, \sqrt{r^2 - \frac{1}{|F'(x_1)|^2}}\right)} \left| B_{2D} \left((x_1, \mathbf{0}, 0), \sqrt{r^2 - |\mathbf{y}_2|^2} \right) \right| d\mathbf{y}_2 \approx \frac{f(x_1 + r)}{|F'(x_1 + r)|^n} (r |F'(x_1 + r)|)^{\frac{n}{2}-1}.$$

To see this, define $\delta = \delta(x_1, r) \in (0, r)$ to be the unique solution to the equation

$$f(x_1 + \sqrt{r^2 - \delta^2}) = \frac{1}{2} f(x_1 + r).$$

Then using that f is increasing and $|F'|$ is decreasing, we have from the mean value theorem the estimate

$$\begin{aligned} f(x_1 + \sqrt{r^2 - \delta^2}) &= f(x_1 + r) - f(x_1 + \sqrt{r^2 - \delta^2}) \\ &= f'(\xi) (r - \sqrt{r^2 - \delta^2}) = |F'(\xi)| f(\xi) (r - \sqrt{r^2 - \delta^2}) \\ &\geq |F'(x_1 + r)| f(x_1 + \sqrt{r^2 - \delta^2}) \frac{\delta^2}{2r}, \end{aligned}$$

which gives an upper bound for δ :

$$\delta \leq \sqrt{\frac{2r}{|F'(x_1 + r)|}}.$$

Similarly we have the estimate

$$\begin{aligned} \frac{1}{2} f(x_1 + r) &= f(x_1 + r) - f(x_1 + \sqrt{r^2 - \delta^2}) \\ &= |F'(\xi)| f(\xi) (r - \sqrt{r^2 - \delta^2}) \\ &\leq |F'(x_1 + \sqrt{r^2 - \delta^2})| f(x_1 + r) \frac{\delta^2}{r}, \end{aligned}$$

which gives the following lower bound for δ ,

$$\delta \geq \sqrt{\frac{r}{2|F'(x_1 + \sqrt{r^2 - \delta^2})|}} \geq \sqrt{\frac{r}{2C|F'(x_1 + r)|}},$$

since

$$(4.2) \quad |F'(x_1 + \sqrt{r^2 - \delta^2})| \leq C |F'(x_1 + r)|.$$

To see this last inequality, note that from the upper bound for δ above, and structure condition (4) in Definition 7, we have

$$\delta^2 \leq \frac{2r}{|F'(x_1 + r)|} \leq \frac{2}{\varepsilon} r (x_1 + r).$$

Now we recall assertion (2) of Lemma 69 in [KoRiSaSh]: if $x_1, x_2 \in (0, R)$ and $\max\left\{\varepsilon x_1, x_1 - \frac{1}{|F'(x_1)|}\right\} \leq x_2 \leq x_1 + \frac{1}{|F'(x_1)|}$, then we have

$$\begin{aligned} |F'(x_1)| &\approx |F'(x_2)|, \\ f(x_1) &\approx f(x_2). \end{aligned}$$

Using this, iterated a fixed number C of times, together with the bound

$$r - \sqrt{r^2 - \delta^2} \approx \frac{\delta^2}{r} \lesssim \frac{1}{|F'(x_1 + r)|},$$

we obtain

$$\left| F' \left(x_1 + r - \frac{C}{|F'(x_1)|} \right) \right| \approx |F'(x_1 + r)|,$$

which proves the inequality (4.2).

We now estimate the integral on the left side of (4.1) separately in two cases. First, if $\delta \leq \sqrt{r^2 - \frac{1}{|F'(x_1)|^2}}$, then we have

$$\begin{aligned}
& \int_{A(0,\delta)} \left| B_{2D} \left((x_1, \mathbf{0}, 0), \sqrt{r^2 - |\mathbf{y}_2|^2} \right) \right| d\mathbf{y}_2 \approx \int_{A(0,\delta)} \frac{f \left(x_1 + \sqrt{r^2 - |\mathbf{y}_2|^2} \right)}{\left| F' \left(x_1 + \sqrt{r^2 - |\mathbf{y}_2|^2} \right) \right|^2} d\mathbf{y}_2 \\
& \approx \int_{A(0,\delta)} \frac{f \left(x_1 + \sqrt{r^2 - |\mathbf{y}_2|^2} \right)}{\left| F' \left(x_1 + \sqrt{r^2 - |\mathbf{y}_2|^2} \right) \right|^2} d\mathbf{y}_2 \approx \delta^{n-2} \frac{f(x_1 + r)}{|F'(x_1 + r)|^2} \\
& \approx \left(\frac{r}{|F'(x_1 + r)|} \right)^{\frac{n}{2}-1} \frac{f(x_1 + r)}{|F'(x_1 + r)|^2} = \frac{f(x_1 + r)}{|F'(x_1 + r)|^n} (r |F'(x_1 + r)|)^{\frac{n}{2}-1}.
\end{aligned}$$

On the other hand, if $\delta > \sqrt{r^2 - \frac{1}{|F'(x_1)|^2}}$, then we have

$$\begin{aligned}
\frac{2r}{|F'(x_1 + r)|} & \geq \delta^2 > r^2 - \frac{1}{|F'(x_1)|^2}, \\
\text{implies } 2r |F'(x_1 + r)| & > r^2 |F'(x_1 + r)|^2 - \frac{|F'(x_1 + r)|^2}{|F'(x_1)|^2}, \\
\text{implies } r^2 |F'(x_1 + r)|^2 & < 2r |F'(x_1 + r)| + \frac{|F'(x_1 + r)|^2}{|F'(x_1)|^2}, \\
\text{implies } (r |F'(x_1 + r)| - 1)^2 & < 1 + \frac{|F'(x_1 + r)|^2}{|F'(x_1)|^2} \\
\text{implies } r |F'(x_1 + r)| & < 1 + \sqrt{1 + \frac{|F'(x_1 + r)|^2}{|F'(x_1)|^2}}.
\end{aligned}$$

Now using $r \geq \frac{1}{|F'(x_1)|}$ in (2) of Proposition 77 in [KoRiSaSh], together with the above, we obtain

$$0 < c \leq r |F'(x_1 + r)| < 3,$$

and so using $r \geq \frac{2}{|F'(x_1)|}$ we have

$$\begin{aligned}
& \int_0^{\sqrt{r^2 - \frac{1}{|F'(x_1)|^2}}} \left| B_{2D} \left((x_1, \mathbf{0}, 0), \sqrt{r^2 - y_2^2} \right) \right| d\mathbf{y}_2 \\
& \approx \sqrt{r^2 - \frac{1}{|F'(x_1)|^2}} \frac{f(x_1 + r)}{|F'(x_1 + r)|^2} \approx r \frac{f(x_1 + r)}{|F'(x_1 + r)|^2} \\
& \approx r |F'(x_1 + r)| \frac{f(x_1 + r)}{|F'(x_1 + r)|^3} \approx \sqrt{r |F'(x_1 + r)|} \frac{f(x_1 + r)}{|F'(x_1 + r)|^3}.
\end{aligned}$$

This finishes the proof of (4.1), and combining the above estimates now completes the proof of Lemma 19 for all $n \geq 3$. \square

Recall from Subsection 1.2 of Chapter 10 of [KoRiSaSh] that in order to obtain a subrepresentation formula involving ∇_A in three dimensions, we defined the cusp-like region $\Gamma(x, r)$ to be the union of a sequence of

‘ends’ $E(x, r_k)$ of balls $B(x, r_k)$, with the sequence $\{r_k\}_{k=1}^\infty$ defined as in (7.1) there, as follows:

$$\begin{aligned} \Gamma(x, r) &= \bigcup_{k=1}^\infty E(x, r_k); \\ E(x, r_k) &\equiv \left\{ y = (y_1, y_2, y_3) : \begin{array}{l} x_1 + r_{k+1} \leq y_1 < x_1 + r_k \\ |y_2| < \sqrt{r_k^2 - (y_1 - x_1)^2} \\ |y_3| < h^*(x_1, y_1 - x_1) \end{array} \right\}. \end{aligned}$$

In the three dimensional case, we obtained a subrepresentation formula in Lemma 110 of [KoRiSaSh] from the equivalence $|E(x, r_k)| \approx |E(x, r_k) \cap B(x, r_k)| \approx |B(x, r_k)|$. We now point out, for future reference, that these estimates and the subrepresentation formula continue to hold if we use the modified ‘end’

$$\tilde{E}(x, r_k) \equiv \left\{ y = (y_1, y_2, y_3) : \begin{array}{l} x_1 + r_{k+1} \leq y_1 < x_1 + r_k \\ |y_2| < \sqrt{r_k^2 - r_{k+1}^2} \\ |y_3| < h^*(x_1, r_k) \end{array} \right\}$$

to define a modified cusp-like region

$$(4.3) \quad \tilde{\Gamma}(x, r) = \bigcup_{k=1}^\infty \tilde{E}(x, r_k).$$

Then using the same argument, we see that Lemma 110 of [KoRiSaSh] extends to higher dimensions with $\tilde{\Gamma}(x, r)$ in place of $\Gamma(x, r)$ in the subrepresentation formula.

Combining this extension of Lemma 110 of [KoRiSaSh] with Lemma 19 above, we obtain that

$$\begin{aligned} K_{B(0, r_0)}(x, y) &= \frac{\hat{d}(x, y)}{|B(x, d(x, y))|} \mathbf{1}_{\tilde{\Gamma}(x, r_0)}(y) \\ &\approx \begin{cases} \frac{1}{r^{n-1} f(x_1)} \mathbf{1}_{\tilde{\Gamma}(x, r_0)}(y), & 0 < r = y_1 - x_1 < \frac{2}{|F'(x_1)|} \\ \frac{|F'(x_1 + r)|^{n-1}}{f(x_1 + r) \lambda(x_1, r)^{n-2}} \mathbf{1}_{\tilde{\Gamma}(x, r_0)}(y), & R \geq r = y_1 - x_1 \geq \frac{2}{|F'(x_1)|} \end{cases}, \end{aligned}$$

where we have defined

$$(4.4) \quad \lambda(x_1, r) \equiv \sqrt{r |F'(x_1 + r)|}.$$

Proposition 20. *Let $n \geq 3$. Assume that for some $\varepsilon > 0$ and $C > 0$ the function*

$$(4.5) \quad \varphi(r) \equiv r^{N+1-\varepsilon} |F'(r)|^N$$

is nondecreasing on $(0, r_0)$. Then:

- (1) *the (Φ, φ) -Sobolev inequality (3.3) holds with geometry F , with φ as in (4.5), and with Φ as in (3.1), $N > 1$,*
- (2) *and if $\varphi_{\max}(r) \equiv \sup_{0 < s < r_0} \varphi(s) < \infty$ is a finite constant function, then the (Φ, φ_{\max}) -Sobolev inequality (3.3) holds with geometry F , with φ as in (4.5), and with Φ as in (3.1), $N > 1$,*
- (3) *and moreover, if we have*

$$(4.6) \quad |F'(r)| \leq C \left(\frac{1}{r} \right)^{1 + \frac{1-\varepsilon}{N}},$$

then the (Φ, φ_{\max}) -Sobolev inequality (3.3) holds with geometry F and $\varphi_{\max}(r) \equiv C$.

Proof. Just as in the proof of Proposition 13, it suffices for Part (1) to prove the analogue of (3.11), i.e.

$$\int_B \Phi \left(\frac{K(x, y) |B|}{\omega(r(B))} \right) d\mu(y) \leq C_N \varphi(r(B)) |F'(r(B))|,$$

for all small balls B of radius $r(B)$ centered at the origin, and where $\omega(r(B))$ is the same as in the proof of Proposition 13, i.e.

$$\omega(r(B)) = \frac{1}{t_N |F'(r(B))|}, \quad t_N > e^{2N}.$$

We have

$$|B(0, r_0)| \approx \frac{f(r_0)}{|F'(r_0)|^n} \lambda(0, r_0)^{n-2},$$

where we recall from (4.4) that

$$\lambda(x_1, r) \equiv \sqrt{r |F'(x_1 + r)|},$$

and we now denote the size of the kernel $K_{B(0, r_0)}(x, y)$ as $\frac{1}{s_{y_1 - x_1}}$ where

$$\frac{1}{s_{y_1 - x_1}} \equiv \begin{cases} \frac{1}{r^{n-1} f(x_1)}, & 0 < r = y_1 - x_1 < \frac{2}{|F'(x_1)|} \\ \frac{|F'(x_1 + r)|^n}{f(x_1 + r) \lambda(x_1, r)^{n-2}}, & 0 < r = y_1 - x_1 \geq \frac{2}{|F'(x_1)|} \end{cases}.$$

We are writing the size of $\frac{1}{K_{B(0, r_0)}(x, y)}$ as $s_{y_1 - x_1} = s_r$ since the quantity s_r can be, roughly speaking, thought of a cross sectional volume analogous to the height h_r in the two dimensional case.

Next, write $\Phi(t)$ as

$$(4.7) \quad \Phi(t) = t\Psi(t), \quad \text{for } t > 0,$$

where for $t \geq E$,

$$\begin{aligned} t\Psi(t) &= \Phi(t) = t(\ln t)^N \\ \implies \Psi(t) &= (\ln t)^N, \end{aligned}$$

and for $t < E$,

$$\begin{aligned} t\Psi(t) &= \Phi(t) = t(\ln E)^N \\ \implies \Psi(t) &= (\ln E)^N. \end{aligned}$$

Now temporarily fix $x = (x_1, \mathbf{x}_2, x_3) \in B_+(0, r_0) \equiv \{x \in B(0, r_0) : x_1 > 0\}$. Using the definition of $\tilde{\Gamma}(x, r_0)$ in (4.3) above, we have for $-x_1 < a < b < r_0 - x_1$ that

$$\begin{aligned} \mathcal{I}_{a,b}(x) &\equiv \int_{\{y \in B_+(0, r_0) : a \leq y_1 - x_1 \leq b\} \cap \tilde{\Gamma}(x, r_0)} \Phi \left(K_{B(0, r_0)}(x, y) \frac{|B(0, r_0)|}{\omega(r_0)} \right) \frac{dy}{|B(0, r_0)|} \\ &= \int_{a+x_1}^{b+x_1} \left[\int_{|\mathbf{x}_2 - \mathbf{y}_2| \leq \sqrt{r_k^2 - r_{k+1}^2}} \left\{ \int_{x_3 - h^*(x_1, r_k)}^{x_3 + h^*(x_1, r_k)} \Phi \left(\frac{1}{s_{y_1 - x_1}} |B(0, r_0)| \frac{|B(0, r_0)|}{\omega(r_0)} \right) dy_3 \right\} dy_2 \right] \frac{dy_1}{|B(0, r_0)|} \\ &= \int_{a+x_1}^{b+x_1} \left[4\sqrt{r_k^2 - r_{k+1}^2} h^*(x_1, r_k) \right] \Phi \left(\frac{1}{s_{y_1 - x_1}} \frac{|B(0, r_0)|}{\omega(r_0)} \right) \frac{dy_1}{|B(0, r_0)|} \\ &\approx \int_{a+x_1}^{b+x_1} s_{y_1 - x_1} \Phi \left(\frac{1}{s_{y_1 - x_1}} \frac{|B(0, r_0)|}{\omega(r_0)} \right) \frac{dy_1}{|B(0, r_0)|} \\ &= \int_{a+x_1}^{b+x_1} s_{y_1 - x_1} \left(\frac{1}{s_{y_1 - x_1}} \frac{|B(0, r_0)|}{\omega(r_0)} \right) \Psi \left(\frac{1}{s_{y_1 - x_1}} \frac{|B(0, r_0)|}{\omega(r_0)} \right) \frac{dy_1}{|B(0, r_0)|}, \end{aligned}$$

where the approximation in the fourth line above comes from the estimates

$$\begin{aligned} 4\sqrt{r_k^2 - r_{k+1}^2} h^*(x_1, r_k) (r_k - r_{k+1}) &= \left| \tilde{E}(x, r_k) \right| \approx |B(x, r_k)| \approx s_{r_k} (r_k - r_{k+1}), \\ s_{r_k} &\approx s_{y_1 - x_1}, \quad \text{for } x_1 + r_{k+1} \leq y_1 < x_1 + r_k. \end{aligned}$$

Thus we have

$$\begin{aligned} \mathcal{I}_{a,b}(x) &\approx \frac{1}{\omega(r_0)} \int_{a+x_1}^{b+x_1} \Psi \left(\frac{1}{s_{y_1 - x_1}} \frac{|B(0, r_0)|}{\omega(r_0)} \right) dy_1 \\ &= \frac{1}{\omega(r_0)} \int_a^b \Psi \left(\frac{1}{s_{y_1 - x_1}} \frac{|B(0, r_0)|}{\omega(r_0)} \right) dr. \end{aligned}$$

and so

$$\begin{aligned}
& \int_{B_+(0, r_0)} \Phi \left(K_{B(0, r_0)}(x, y) \frac{|B(0, r_0)|}{\omega(r_0)} \right) \frac{dy}{|B(0, r_0)|} \\
&= \mathcal{I}_{-x_1, r_0-x_1}(x) \\
&= \frac{2}{\omega(r_0)} \int_{-x_1}^{r_0-x_1} \Psi \left(\frac{1}{s_r} \frac{|B(0, r_0)|}{\omega(r_0)} \right) dr.
\end{aligned}$$

Just as in the two-dimensional proof above, it suffices to prove

$$(4.8) \quad \mathcal{I}_{0, r_0-x_1} = \frac{1}{\omega(r_0)} \int_0^{r_0-x_1} \Psi \left(\frac{1}{s_r} \frac{|B(0, r_0)|}{\omega(r_0)} \right) dr \leq C_N \varphi(r_0) |F'(r_0)|,$$

where $|B(0, r_0)|$ is now the Lebesgue measure of the n -dimensional ball $B(0, r_0) = B_{nD}(0, r_0)$.

To prove this we divide the interval $(0, r_0 - x_1)$ of integration in r into three regions as before:

(1): the small region \mathcal{S} where $\frac{|B(0, r_0)|}{s_r \varphi(r_0)} \leq E$,

(2): the big region \mathcal{R}_1 that is disjoint from \mathcal{S} and where $r = y_1 - x_1 < \frac{2}{|F'(x_1)|}$ and

(3): the big region \mathcal{R}_2 that is disjoint from \mathcal{S} and where $r = y_1 - x_1 \geq \frac{2}{|F'(x_1)|}$.

The region \mathcal{S} is handled just as before.

We now turn to the first big region \mathcal{R}_1 where we have $s_{y_1-x_1} \approx r^{n-1} f(x_1)$. The condition that \mathcal{R}_1 is disjoint from \mathcal{S} gives

$$\begin{aligned}
\frac{|B(0, r_0)|}{r^{n-1} f(x_1) \omega(r_0)} &> E, \quad \text{i.e. } r < \left(\frac{A}{E} \right)^{\frac{1}{n-1}}; \\
\text{where } A &= A(x_1) \equiv \frac{|B(0, r_0)|}{f(x_1) \omega(r_0)},
\end{aligned}$$

and so as before

$$\int_{\mathcal{R}_1} \Phi \left(K_{B(0, r_0)}(x, y) \frac{|B(0, r_0)|}{\omega(r_0)} \right) \frac{dy}{|B(0, r_0)|} = \frac{1}{\omega(r_0)} \int_0^{\min\left\{\left(\frac{A}{E}\right)^{\frac{1}{n-1}}, \frac{2}{|F'(x_1)|}\right\}} \left(\frac{A}{r^{n-1}} \right)^{\psi\left(\frac{A}{r^{n-1}}\right)} dr.$$

We will now show that

$$\frac{1}{\omega(r_0)} \int_0^{\min\left\{\left(\frac{A}{E}\right)^{\frac{1}{n-1}}, \frac{2}{|F'(x_1)|}\right\}} \left[\ln \left(\frac{A}{r^{n-1}} \right) \right]^N dr \leq C_N \varphi(r_0) |F'(r_0)|.$$

Now if $\left(\frac{A}{E}\right)^{\frac{1}{n-1}} \leq \frac{2}{|F'(x_1)|}$, then with the change of variable $t = \frac{A}{r^{n-1}}$,

$$\begin{aligned}
& \frac{1}{\omega(r_0)} \int_0^{\min\left\{\left(\frac{A}{E}\right)^{\frac{1}{n-1}}, \frac{2}{|F'(x_1)|}\right\}} \left[\ln \left(\frac{A}{r^{n-1}} \right) \right]^N dr = C \frac{1}{\omega(r_0)} A^{\frac{1}{n-1}} \int_E^\infty (\ln t)^N \frac{dt}{t^{\frac{n}{n-1}}} \\
& \leq C_N \frac{1}{\omega(r_0)} \frac{1}{|F'(x_1)|} = C_N \frac{|F'(r_0)|}{t_N} \frac{1}{|F'(x_1)|} \leq C'_N,
\end{aligned}$$

This proves (4.8) if $\left(\frac{A}{E}\right)^{\frac{1}{n-1}} \leq \frac{1}{|F'(x_1)|}$ since $\varphi(r_0) |F'(r_0)| \geq 1$.

So we now suppose that $\left(\frac{A}{E}\right)^{\frac{1}{n-1}} > \frac{1}{|F'(x_1)|}$. Making a change of variables

$$R = \frac{A}{r^{n-1}} = \frac{A(x_1)}{r^{n-1}}$$

we obtain

$$\frac{1}{\omega(r_0)} \int_0^{\frac{2}{|F'(x_1)|}} \Psi \left(\frac{A}{r^{n-1}} \right) dr = \frac{1}{\omega(r_0)} A^{\frac{1}{n-1}} \int_{\frac{1}{2^{n-1} A |F'(x_1)|^{n-1}}}^\infty (\ln R)^N R^{-\frac{n}{n-1}} dR.$$

Integrating by parts gives as before

$$\begin{aligned} \int_{\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}}}^{\infty} (\ln R)^N R^{-\frac{n}{n-1}} dR &= \int_{\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}}}^{\infty} (\ln R)^N R \left(-\frac{n-1}{n} R^{-\frac{n}{n-1}} \right)' dR \\ &= \frac{n-1}{n} \frac{\left(\ln \left(\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}} \right) \right)^N \left(\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}} \right)}{\left(\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}} \right)^{\frac{n}{n-1}}} \\ &\quad + \frac{n-1}{n} \left(1 + \frac{N}{\ln E} \right) \int_{\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}}}^{\infty} (\ln R)^N R^{-\frac{n}{n-1}} dR. \end{aligned}$$

Taking E large enough depending on N we can assure

$$\frac{n-1}{n} \left(1 + \frac{N}{\ln E} \right) \leq \frac{3}{4},$$

which, after absorbing the second term on the right, gives

$$\int_{\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}}}^{\infty} (\ln R)^N R^{-\frac{n}{n-1}} dR \lesssim \frac{\left(\ln \left(\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}} \right) \right)^N \left(\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}} \right)}{\left(\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}} \right)^{\frac{n}{n-1}}}.$$

Therefore

$$\begin{aligned} \mathcal{I}_{0, \frac{2}{|F'(x_1)|}}(x) &= \frac{1}{\omega(r_0)} A^{\frac{1}{n-1}} \int_{\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}}}^{\infty} (\ln R)^N R^{-\frac{n}{n-1}} dR \\ &\lesssim \frac{1}{\omega(r_0)} A^{\frac{1}{n-1}} \frac{\left(\ln \left(\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}} \right) \right)^N}{\left(\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}} \right)^{\frac{1}{n-1}}} \\ &\lesssim \frac{1}{\omega(r_0) |F'(x_1)|} \left(\ln \left(\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}} \right) \right)^N \\ &\equiv \frac{1}{\omega(r_0)} \mathcal{F}(x_1). \end{aligned}$$

We now wish to find the maximum of the function $\mathcal{F}(x_1)$ given by

$$\begin{aligned} \mathcal{F}(x_1) &\equiv \frac{1}{|F'(x_1)|} \left(\ln \left(\frac{1}{2^{n-1}A|F'(x_1)|^{n-1}} \right) \right)^N \\ &= \frac{1}{|F'(x_1)|} \left(\ln \left(\frac{1}{2^{n-1}} \frac{|B(0, r_0)| |F'(x_1)|^{n-1}}{\omega(r_0) f(x_1)} \right) \right)^N \\ &= \frac{1}{|F'(x_1)|} \left(\ln \left(c(r_0) \frac{|F'(x_1)|^{n-1}}{f(x_1)} \right) \right)^N, \end{aligned}$$

where

$$c(r_0) = \frac{1}{2^{n-1}} \frac{|B(0, r_0)|}{\omega(r_0)}.$$

This turns out to be at most $\varphi(r_0)$ as we now demonstrate. We rewrite

$$\mathcal{F}(x_1) = \frac{1}{|F'(x_1)|} (\ln c(r_0) + (n-1) \ln |F'(x_1)| + F(x_1))^N.$$

Proceeding the same way as in the proof of Proposition 13, we obtain that at x_1^* maximizing $\mathcal{F}(x_1)$, we have

$$\mathcal{F}(x_1^*) = \frac{N^N}{|F'(x_1^*)|} \left(1 + (n-1) \frac{|F'(x_1^*)|^2}{F''(x_1^*)} \right)^N,$$

and satisfies

$$\mathcal{F}(x_1^*) \leq C_N (x_1^*)^N |F'(x_1^*)|^{N-1} \lesssim \varphi(x_1^*) \leq \varphi(r_0),$$

by properties (5) and (4) on the geometry F , and the monotonicity assumption (4.5).

We now show that for the integral over the second region \mathcal{R}_2 , we get a similar condition with an extra power of $1 + \varepsilon$ for some $\varepsilon > 0$ in the definition of $\mathcal{F}(x_1)$. Let $r \geq \frac{2}{|F'(x_1)|}$. We then have

$$|B(0, r_0)| = \frac{f(r_0)\lambda(0, r_0)^{n-2}}{|F'(r_0)|^n} \text{ and } \frac{1}{s_{y_1-x_1}} \approx \frac{|F'(x_1+r)|^n}{f(x_1+r)\lambda(x_1, r)^{n-2}},$$

and the integral $\frac{1}{\omega(r_0)} \int_{x_1 + \frac{2}{|F'(x_1)|}}^{r_0} \left[\ln \left(\frac{|B(0, r_0)|}{s_r \omega(r_0)} \right) \right]^N dr$ to be estimated is

$$\begin{aligned} I_{\mathcal{R}_2} &\equiv \frac{1}{\omega(r_0)} \int_{x_1 + \frac{2}{|F'(x_1)|}}^{r_0} \left[\ln \left(\frac{|B(0, r_0)|}{\omega(r_0)} \frac{|F'(y_1)|^n}{f(y_1)\lambda(x_1, y_1 - x_1)^{n-2}} \right) \right]^N dy_1 \\ &= \frac{1}{\omega(r_0)} \int_{x_1 + \frac{2}{|F'(x_1)|}}^{r_0} \left[\ln \left(c(r_0) \frac{|F'(y_1)|^n}{f(y_1)\lambda(x_1, y_1 - x_1)^{n-2}} \right) \right]^N dy_1, \end{aligned}$$

where $c(r_0) = \frac{|B(0, r_0)|}{\omega(r_0)}$.

Again, we would like to estimate the above integral by $C_N \varphi(r_0) |F'(r_0)|$. Write

$$\begin{aligned} &\int_{x_1 + \frac{1}{|F'(x_1)|}}^{r_0} \left(\ln \left(c(r_0) \frac{|F'(y_1)|^n}{f(y_1)\lambda(x_1, y_1 - x_1)^{n-2}} \right) \right)^N dy_1 \\ &= \int_{y_1 + \frac{1}{|F'(y_1)|}}^{r_0} y_1^{1-\varepsilon} \left(\ln \left(c(r_0) \frac{|F'(y_1)|^n}{f(y_1)\lambda(x_1, y_1 - x_1)^{n-2}} \right) \right)^N \frac{dy_1}{y_1^{1-\varepsilon}}, \end{aligned}$$

and define

$$(4.9) \quad \mathcal{G}_{x_1}(y_1) \equiv y_1^{1-\varepsilon} (\ln c(r_0) + n \ln |F'(y_1)| + F(y_1) - (n-2) \ln \lambda(x_1, y_1 - x_1))^N.$$

Now we look for the maximum of the function $\mathcal{G}_{x_1}(y_1)$ on $(x_1 + \frac{1}{|F'(x_1)|}, x_1 + r_0)$. Recalling that $\lambda(x_1, y_1) = \sqrt{(y_1 - x_1) |F'(y_1)|}$, we obtain

$$\begin{aligned} \mathcal{G}_{x_1}(y_1) &= y_1^{1-\varepsilon} \left(\ln c(r_0) + n \ln |F'(y_1)| - \frac{n-2}{2} \ln(y_1 - x_1) - \frac{n-2}{2} \ln |F'(y_1)| + F(y_1) \right)^N \\ &= y_1^{1-\varepsilon} \left(\ln c(r_0) + \frac{n+2}{2} \ln |F'(y_1)| + F(y_1) + \frac{n-2}{2} \ln \frac{1}{y_1 - x_1} \right)^N. \end{aligned}$$

Now recall the condition

$$y_1 - x_1 = r \geq \frac{2}{|F'(x_1)|}.$$

We claim that for $y_1 \in \left(x_1 + \frac{1}{|F'(x_1)|}, r_0\right)$ we have

$$y_1 - x_1 \geq \frac{c}{|F'(y_1)|}$$

for some $0 < c \leq 1$. Indeed, if $y_1 \geq 2x_1$ we have

$$y_1 - x_1 \geq \frac{y_1}{2} \geq \frac{c}{|F'(y_1)|}$$

by structure condition (4) in Definition 7. On the other hand, if $x_1 \leq y_1 \leq 2x_1$, we have using structure condition (3) that $|F'|$ is doubling and that

$$y_1 - x_1 \geq \frac{2}{|F'(x_1)|} \approx \frac{1}{|F'(y_1)|}.$$

Altogether this permits us to bound \mathcal{G}_{x_1} by an expression $\tilde{\mathcal{G}}$ that is *independent* of x_1 :

$$\begin{aligned} \mathcal{G}_{x_1}(y_1) &\leq y_1^{1-\varepsilon} \left(\ln c(r_0) + \ln \frac{1}{c} + \frac{n+2}{2} \ln |F'(y_1)| + F(y_1) + \frac{n-2}{2} \ln |F'(y_1)| \right)^N \\ &\leq y_1^{1-\varepsilon} \left(\ln c(r_0) + \ln \frac{1}{c} + n \ln |F'(y_1)| + F(y_1) \right)^N \equiv \tilde{\mathcal{G}}(y_1). \end{aligned}$$

Now we look for the maximum of the function $\tilde{\mathcal{G}}(y_1)$ above on $\left(x_1 + \frac{1}{|F'(x_1)|}, r_0\right)$. Just as in the proof of Proposition 13 above, we obtain by differentiating $\tilde{\mathcal{G}}(y_1)$ and setting the derivative equal to zero, the following implicit expression for y_1^* maximizing $\tilde{\mathcal{G}}(y_1)$:

$$\begin{aligned} (1-\varepsilon)(y_1^*)^{-\varepsilon} \left(\ln c(r_0) + \ln \frac{1}{c} + n \ln |F'(y_1^*)| + F(y_1^*) \right)^N \\ + N(y_1^*)^{1-\varepsilon} \left(\ln c(r_0) + \ln \frac{1}{c} + n \ln |F'(y_1^*)| + F(y_1^*) \right)^{N-1} \left(-n \frac{F''(y_1^*)}{|F'(y_1^*)|} - |F'(y_1^*)| \right) = 0, \end{aligned}$$

which gives

$$\left(\ln c(r_0) + \ln \frac{1}{c} + n \ln |F'(y_1^*)| + F(y_1^*) \right) = \frac{N}{1-\varepsilon} y_1^* \left(n \frac{F''(y_1^*)}{|F'(y_1^*)|} + |F'(y_1^*)| \right).$$

Thus we have

$$\begin{aligned} \tilde{\mathcal{G}}(y_1^*) &= (y_1^*)^{1-\varepsilon} \left(\ln c(r_0) + \ln \frac{1}{c} + n \ln |F'(y_1^*)| + F(y_1^*) \right)^N \\ &= (y_1^*)^{N+1-\varepsilon} \left(\frac{N}{1-\varepsilon} \left(n \frac{F''(y_1^*)}{|F'(y_1^*)|} + |F'(y_1^*)| \right) \right)^N \\ &\lesssim (y_1^*)^{N+1-\varepsilon} |F'(y_1^*)|^N, \end{aligned}$$

which is an increasing function by assumption (4.5), and satisfies

$$\mathcal{G}(y_1) \leq \tilde{\mathcal{G}}(y_1^*) \leq \varphi(r_0).$$

Parts (2) and (3) follow as in the proof of Proposition 13, and this concludes the proof of Proposition 20. \square

4.1. Higher dimensional local boundedness. In exactly the same way as we derived Corollaries 14, 15 and 16 above, we can obtain their higher dimensional counterparts, culminating in a sharp local boundedness theorem for $n \geq 3$ in Corollary 23 below, which completes the proof of Theorem 1.

Corollary 21. *The Orlicz-Sobolev bump inequality with $\Phi = \Phi_N$ holds for the geometry D_σ if $0 < \sigma < \frac{1}{N}$.*

Corollary 22. *The Orlicz-Sobolev bump inequality with $\Phi = \Phi_N$ holds for the geometry F provided $F \leq D_\sigma$ for some $0 < \sigma < \frac{1}{N}$.*

Corollary 23. *Weak subsolutions to $\mathcal{L}u = \phi$ are locally bounded for the geometry F provided $F \leq D_\sigma$ for some $0 < \sigma < 1$. Moreover, if $n \geq 3$ and $\sigma > 1$, then there exists an unbounded weak solution u in a neighbourhood of the origin in \mathbb{R}^n to the equation $\mathcal{L}u = 0$ with geometry $F = D_\sigma$.*

Proof. The first assertion follows from the previous corollary and Proposition 12. The second assertion follows from Theorem 115 of [KoRiSaSh], which extends a counterexample of Morimoto [Mor] to our setting, and which was in turn based on a sharpness result of Kusuoka-Strook in [KuStr]. \square

4.2. Higher dimensional maximum principle. We have the following higher dimensional analogue of Theorem 17, which completes the proof of Theorem 2.

Theorem 24. *Let Ω be a domain in \mathbb{R}^n and let F be any geometry satisfying the structure conditions in Definition 7. Assume that u is a weak subsolution to $\mathcal{L}u = \phi$ in Ω with A -admissible ϕ , and that u is nonpositive in the weak sense on the boundary $\partial\Omega$. Then the following maximum principle holds,*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C \|\phi\|_{X(\Omega)},$$

and in particular u is globally bounded.

Proof. The proof is identical to that of the two-dimensional Theorem 17 above if we replace the height h_r in that proof with the ‘cross sectional area’ s_r that appears in the proof of Proposition 20 above. \square

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